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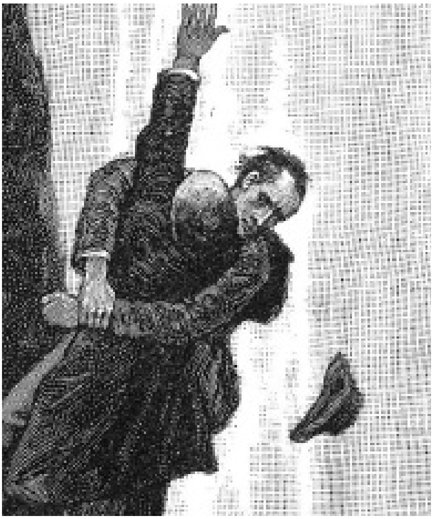

Victorian illustration of Holmes and Moriarty at the Reichenbach Falls. Inside, T. Dence examines the mathematical writings featured in the movie Sherlock Holmes: A Game of Shadows.

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THE MATHEMATICAL GAZETTE

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Cover Picture - Alain Goriely and Derek Moulton devised some mathematical writings for placement on Moriarty's blackboard in the movie Sherlock Holmes: A Game of Shadows. The article features in this promotional issue on page 7

R. L. Goodstein and mathematical logic

GRAHAM HOARE

Born in London, Reuben Louis Goodstein (1912-1985) completed his secondary education at St Paul's School and in 1931 proceeded to Magdalene College, Cambridge, with a Major Open Scholarship to read mathematics. He graduated in 1933 having taken firsts in Parts I and II of the Mathematical Tripos. From 1933 to 1935 his research on transfinite numbers was supervised by Professor J. E. Littlewood. He took a MSc and left Cambridge in 1935 to take up an appointment as lecturer in pure and applied mathematics at Reading University, a position he held until late 1947. While undertaking a strenuous teaching load at Reading his research interests were developing and for this work he received a PhD from the University of London in 1946, which was supervised by the philosopher Ludwig Wittgenstein.

In 1948 Goodstein was appointed Professor and head of department at University College, Leicester, a position he held until his retirement in 1977. Remarkably, he was the first person whose main interests were in mathematical logic to hold a chair in a British university. At the critical time of transition from college to full university at Leicester in 1957 Goodstein was Dean of Science and from 1966 to 1969 he was Pro-Vice-Chancellor. By the time he retired his department had expanded from a staff of six to twenty-three with a corresponding increase in student numbers.

Towards the end of the 19th century something of a crisis had arisen in the foundations of mathematics, primarily in the theory of sets. Cantor himself, in 1899, discovered a paradoxical result and Russell found a serious flaw in Frege's work. These difficulties elicited various responses, principally from Russell, Brouwer, Zermelo (and Fraenkel) and Skolem. The axiomatic set-theoretic system of Zermelo-Fraenkel was the most widely adopted but this came at a price; it cannot be proved to be consistent. As André Weil was alleged to have said, "God exists since mathematics is consistent and the Devil exists since we cannot prove it." Essentially, there are two opposing trends in the study of the foundations of mathematics, namely, the infinitistic or set-theoretical and the finitistic or arithmetical; Goodstein adhered to the latter. Indeed, not even Brouwer's intuitionist approach was sufficiently stringent for him. He rejected the *tertium non datur* (law of excluded middle), $\text{not}\,(\forall x)\,R(x) \Rightarrow (\exists x)\,\text{not}\,R(x)$ *and* its analogue $\text{not}\,(\exists x)\,R(x) \Rightarrow (\forall x)\,\text{not}\,R(x)$ and was led to a complete rejection of quantification theory.

It was Thoralf Skolem, in a 1923 paper, who grasped clearly and decisively the full power of the recursive mode of thought in his formulation of a portion of elementary arithmetic. Skolem proposed that arithmetic should be based not on set theory but on recursion, a modest proposal since it sought to establish a new foundation for arithmetic and not the whole of mathematics. Goodstein's extreme finitist view of mathematics led him to investigate those concepts and theorems from arithmetic and analysis which can be interpreted *primitive* recursively. He formulated primitive recursive arithmetic (PRA) as a logic-free equation calculus; all propositions are equations of the form A = B where A and B are primitive recursive functions or terms which conform to standard substitution and uniqueness rules. The propositional connectives (and, not, or, implies) and the bounded quantifiers $(\forall, \exists)$ are introduced arithmetically. In this axiom-free equation calculus the principle of mathematical induction is superfluous. Influenced by Wittgenstein, whose classes he attended during his time at Cambridge, and encouraged by Paul Bernays, Goodstein developed his ideas of PRA and analysis in a series of monographs (particularly [1, 2]) and research papers (beginning with [3]). In analysis, for example, he gave the theory of exponential, logarithmic and circular functions from his strictly finitistic viewpoint and after quite a struggle showed how Gauss's second proof of the fundamental theorem of algebra could be rewritten in finitist form. Much of classical analysis can be saved, then, but Weierstrass's theorem, that all monotone bounded sequences have a limit, is lost, since unbounded quantifiers are forbidden. Goodstein's attitude to the foundations was not widely shared in Britain but the Leningrad (St Petersburg) school showed interest and three of his books received Russian translations.

In his first incompleteness theorem Gödel constructed a sentence which is undecidable by the methods of PA (Peano Arithmetic; Peano's axioms augmented by first-order predicate logic). Since 1931 mathematicians had been searching for a strictly mathematical example of a theorem that is true but not provable by the methods of PA. The first such example was found in 1977 by Leo Harrington and Jeff Paris which involved a simple extension of the Finite Ramsey theorem, a result in combinatorics which grew out of the work of Laurie Kirby and Paris. Perhaps the most astonishing result of this genre is Goodstein's theorem, *The restricted ordinal theorem*, which involves a highly counter-intuitive result in number theory. It begins by expressing a positive integer in some base, x, say, and also writing each exponent in the same base x. In base 2, for example: $43 = 2^{2^2+1} + 2^{2+1} + 2 + 1$. A *Goodstein sequence* is now defined by repeating the following process which involves replacing each occurrence of the base x by $x + 1$ and then subtracting 1 so the next number in our example becomes $3^{3^3+1} + 3^{3+1} + 3$ and the next, $4^{4^4+1} + 4^{4+1} + 3$, and so on. Goodstein's theorem states that such sequences terminate at zero after finitely many steps. According to Kirby and Paris, even if we start with $4 = 2^2$ the process reaches zero at base $3.2^{402,653,211} - 1$.

Goodstein published his proof of the theorem in 1944 using transfinite induction (ε_0-induction) for ordinals less than ε_0 (i.e. the least of the solutions for ε to satisfy $\varepsilon = \omega^\varepsilon$, where ω is the first transfinite ordinal) and he noted the connection with Gentzen's proof of the consistency of arithmetic by the same means. The significance of Goodstein's theorem emerged in 1982 when Kirby and Paris proved that it is unprovable in PA but can be proved in a stronger system such as second order arithmetic. It seems strange that one of Goodstein's most important results should run counter to his efforts to reconstruct mathematics along finitist lines. André Weil considered Gentzen a lunatic who used transfinite induction to prove the consistency of ordinary induction. Goodstein took the view that Weil's criticism overlooks the essential point that what Gentzen achieved was an elimination of quantifiers at the cost of introducing a principle of transfinite induction. Goodstein described transfinite induction as a 'minimum deviation from the previously accepted field of finitist processes'.

According to Ray Monk, in his splendid biography of Wittgenstein, Wittgenstein's five favourite students were '(Francis) Skinner, Louis Goodstein, H. S. M. Coxeter, Margaret Masterman and Alice Ambrose'. Skinner, as a contemporary of Goodstein's at St Paul's, was allocated the task of recording Wittgenstein's deliberations during the classes. Copies were made and distributed but Skinner, now a wrangler, died suddenly in 1941, and his detailed notes of Wittgenstein's lectures were parcelled up by Wittgenstein and sent to Goodstein at Reading. Eventually, as Goodstein approached retirement, he chose to donate these manuscripts to the MA Library (at Leicester). This body of work comprised much of what became known as the *Blue* and *Brown* books. These underpinned Wittgenstein's second great work after the *Tractatus*, his *Philosophical Investigations*, which was published posthumously through the labours of philosophers working from sources parallel to the dormant manuscripts held privately for some thirty years by Goodstein.

Now a diversion. Goodstein and Alan Turing were born in the same year and both entered Cambridge in 1931. According to letters written home to his mother Sara in October and November 1937 while he was in America, Turing received a substantial paper from the LMS secretary to referee and, later, a letter from the secretary of the Faculty Board of Mathematics at Cambridge asking if he would be a PhD examiner. The candidate was Goodstein in both cases. Turing's response to the paper in his own words was that "The author's technique was hopelessly faulty, and his work after about page 30 was based on so many erroneous notions as to be quite hopeless". The dissertation, which involved a development of recursive analysis, including measure theory, was largely in error because the ambiguity of the representation of real numbers as binary decimals had been overlooked. It is all the more laudable that after this inauspicious start to his mathematical research Goodstein recovered to be a distinguished mathematician and teacher.

As has been well documented Goodstein was especially active in the MA. He contributed some 70 notes and articles as well as hundreds of reviews to the *Mathematical Gazette* which he sustained at a high academic level during his editorship from 1956 to 1962. He was President of the Association in 1975-76.

Goodstein wrote extensively and with great clarity. He was especially adept at explaining difficult ideas. His *Recursive number theory* [1], for example, is a complete introduction to early researches on recursive functions. His 1971 *Development of mathematical logic* is an excellent overview of mathematical logic. His *Gazette* article, The Decision Problem, (February 1957), is lucid and precise and his Presidential Address of April 1976 is a cogent summary of his research preoccupations. He was, however, disappointed that his 1948 textbook, *Mathematical Analysis*, which incorporated a distinctive approach to the 'uniform calculus', failed to impress.

By all accounts Goodstein was an effective and enthusiastic teacher. Many British logicians were influenced by him or had worked with him or did research under his guidance. Many of his research students took up posts in higher education. Clearly the mathematical world has been enriched by his labours and the MA will be ever in his debt.

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1. R. L. Goodstein, *Recursive number theory*, North-Holland, Amsterdam (1957); Russian edition (1970).
2. R. L. Goodstein, *Recursive analysis*, North-Holland, Amsterdam (1961); Russian edition (1970).
3. R. L. Goodstein, Function theory in an axiom-free equation calculus, *Proc. London Math. Soc*. (1945) pp. 401-434.

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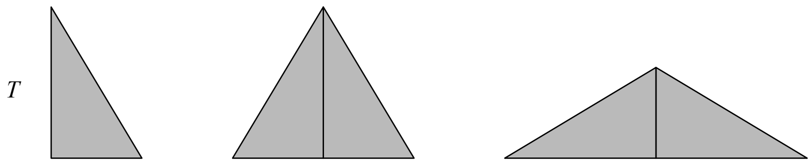
Problem 1.

The diagram shows a right-angled triangle T and two isosceles triangles, each formed from two copies of T.



Let T have inradius r and let the two isosceles triangles have inradii r_1 and r_2. (The inradius is the radius of the largest circle that can be drawn inside the triangle.)

Find and prove a simple relationship between r, r_1 and r_2.

See the solution to the problem at journals.cambridge.org/trial_MAG

Holmes + Moriarty = Mathematics

THOMAS DENCE

Introduction

In the autumn of 2011 the movie *Sherlock Holmes: A Game of Shadows* appeared (see [1]). It was a sequel to the Holmes movie from two years earlier. Unlike the first movie, *A Game of Shadows* Holmes' arch enemy Professor Moriarty featured more prominently than Holmes himself.

One scene, in particular, has Holmes sizing up his opponent by making a visit to Moriarty at his university office. But in the background sits Moriarty's blackboard, filled completely with mathematical equations, symbols, drawings, and formulas. One of these deserves our special attention.

Background

According to the literature on Sir Arthur Conan Doyle we find James Nolan Moriarty born in 1849 in Ireland, educated at home, and from early on possessed remarkable mathematical skills. His graduation from University College in Dublin in 1871 coincided with the publishing of his famous mathematical work, *A Treatise on the Binomial Theorem*, which enhanced his reputation and garnered him a teaching position at a small university in Durham in northern England. Six years later he was, rather surreptitiously, asked to leave with one probable reason being his growing association with the more extreme elements in the Irish nationalist movement as this faction sought to attain independence from British rule. Moriarty moved to London in 1878 and found meagre employment as a mathematics tutor to young British soldiers. This was a blow to Moriarty's ego as a mathematician. Consequently his attention was refocused on the Irish attempts to win independence by any means, legal or illegal. Here, his mathematical skills were put to use by the Irish Republican Brotherhood, known as the Fenians, because Moriarty became a creator of ciphers and a breaker of codes [2]. The codes that the Fenians had been using were simple Caesar cipher codes where letters were interchanged with others located a fixed distance away in the alphabet. Unfortunately for the Irish these codes were commonly broken by the British police. Under Moriarty's mathematical super-vision, though, the new codes became fiendishly complex.

Sometime around the later 1870s Sherlock Holmes turned his attention to helping England investigate the widespread bombings and assassinations that were presumably instigated by Irish nationalists. Apart from his private cases involving, for instance, the 'five orange pips' and the 'speckled band', we find in *The Adventure of the Bruce-Partington Plans* an episode dealing with national security [3]. For the most part, though, the narratives of Holmes' adventures, as written by his famous colleague Dr. Watson, failed to mention much about his work in helping the British government's continual battle against Irish nationalists. Holmes' brother Mycroft had been studying the political scene for a number of years and was influential in helping Sherlock get to the root of some of the terrorist's activity. We find then, in late 1882, that the Holmes brothers first encounter the name Moriarty while investigating the intricate politics of the Irish nationalists.

Holmes viewed himself as an expert on ciphers and codes. In *The Adventure of the Dancing Men*, a story that featured messages in code, Holmes says to Watson, "I am fairly familiar with all forms of secret writing, and am myself the author of a trifling monograph upon the subject, in which I analyse 160 separate ciphers." Thus we find Holmes spending considerable time trying to decode the communication of the Irish nationalists that the police had intercepted. This proved difficult and Holmes began to realise that the ciphers were the work of a mind of a different calibre to those that had produced earlier codes.

By 1885 Holmes is obsessed with Moriarty, and has no doubt that he is responsible for many of the bombings and killings that have happened in the London area. To his friend Watson, Holmes utters the famous descriptive lines, "He is the Napoleon of crime. He is the organiser of half that is evil and of nearly all that is undetected in this great city." Yet even as late as 1888 Holmes had never met Moriarty.

But in *A Game of Shadows* we witness Holmes finally paying a visit to Moriarty in his university office, which really cannot be because Moriarty no longer holds university status.

The Blackboard

The folks at Warner Brothers, who made the movie, wanted some sense of authenticity to these scenes. They contacted OCCAM (the Oxford Centre for Collaborative Applied Mathematics) and asked for mathematical help [4]. They hired Professors Alain Goriely and Derek Moulton to devise some mathematical writings for placement on Moriarty's blackboard that would satisfy several criteria. It had to be mathematics that would be pertinent for the time, and would be consistent with what Moriarty would have known and would have worked on and used. The mathematics chosen was essentially from two different areas.

On one hand the board contains numerous symbols, text, and partial differential equations inherent to the study of celestial mechanics, in particular the n-body problem. This problem seeks to understand the interaction, due to gravity, of n bodies of mass in space. The rationale behind this is that in *The Valley of Fear* Holmes mentions that Moriarty authored the book *The Dynamics of an Asteroid*, and that the book was of such high quality that "no man in the scientific press was capable of criticising it." In the top left corner of the blackboard Goriely and Moulton have sketched two circles, one representing Earth and the other representing the Sun, with 'comets' written between them (Figure 1).

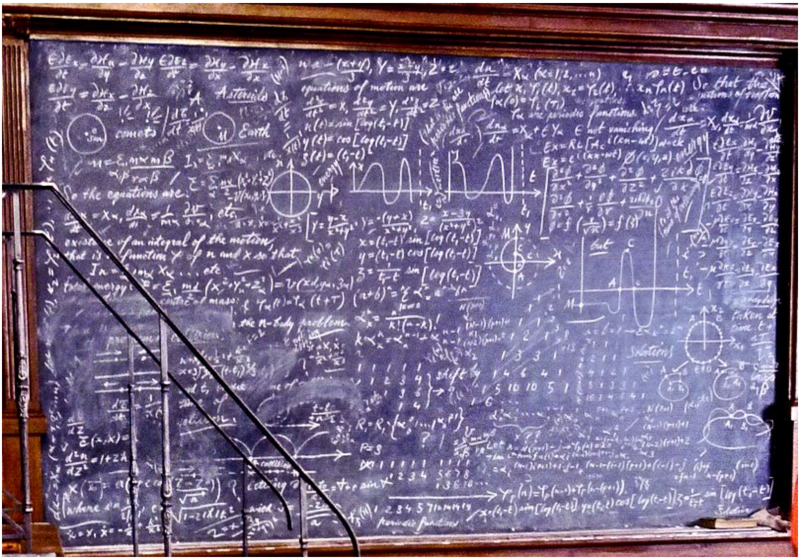


FIGURE 1

The other mathematical area of interest on the board involves one of Moriarty's schemes for coding messages. Goriely and Moulton decided to connect this scheme with Pascal's triangle (also located on the board) which was appropriate since Moriarty had written *A Treatise on the Binomial Theorem*. Located under Pascal's triangle on the board is the recursively defined Fibonacci-type sequence

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

with $F_p(n) = 1$ if $n \leqslant 1$, and the recursion holds for all $n > 1$. The integer p is fixed, and stated initially, and represents a public key for the coding scheme. In particular, for $p = 0, 1, 2, 3$ and $n > 1$ we get the following sequences:

$p = 0$	1, 2, 4, 8, 16, 32, 64, …
$p = 1$	1, 2, 3, 5, 8, 13, 21, 34, …
$p = 2$	1, 2, 3, 4, 6, 9, 13, 19, …
$p = 3$	1, 2, 3, 4, 5, 7, 10, 14, …

and we note here that the first sequence ($p = 0$) is merely the sum of the terms in each row of Pascal's triangle. The second sequence ($p = 1$), which is the Fibonacci sequence, is the sum of the terms on a first diagonal of Pascal's triangle, as shown in Figure 3a.

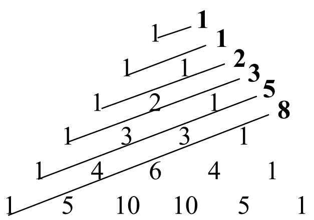


FIGURE 3a

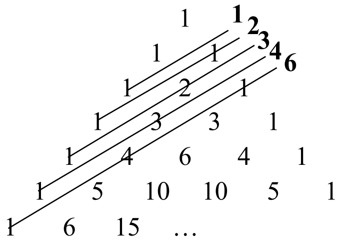


FIGURE 3b

The third sequence ($p = 2$) then represents the sum of the terms on a second diagonal of Pascal's triangle (Figure 3b), and the pattern continues for $p \geqslant 3$. We can be a little more precise in describing how these diagonals are constructed. First, we indicate that adjacent rows in the triangle are one unit apart, and that adjacent terms in each row are two units apart. For each given p, each new diagonal starts with the leading 1 in each row of the triangle. From that 1 we move $p + 2$ units to the right, followed by p units up, and if that lands on a term in the triangle then that term gets added to the 1; then we move another $p + 2$ units to the right and p units up, and add again if we land on a term from the triangle. The pattern continues.

The mathematical link that provides an integral component to Moriarty's code is stated as follows [4].

Theorem: Let $p \geqslant 0$ and $F_p(n)$ be defined as above. Given any integer $N > 1$ there exist unique non-negative integers n, m such that $N = F_p(n) + m$, with $m < F_p(n - p)$.

Proof:

(existence)

case 1: Suppose $N = F_p(n)$ for some n. Then let $m = 0$ and $m < F_p(n - p)$.

case 2: Let $n \geqslant 1$ be such that $F_p(n) < N < F_p(n + 1)$. Set $m = N - F_p(n)$. Then

$$N - F_p(n) < F_p(n + 1) - F_p(n) = F_p(n + 1 - (p + 1)) = F_p(n - p).$$

(uniqueness)

case 1: Assume $N = F_p(n)$ for some n, and suppose also that $N = F_p(n - k) + m$ with $m < F_p(n - k - p)$ and $k \geqslant 1$. Subtracting gives

$$\begin{aligned} m &= F_p(n) - F_p(n - k) \\ &= \left[F_p(n - 1) + F_p(n - 1 - p)\right] - F_p(n - k) \\ &= \left[F_p(n - 2) + F_p(n - 2 - p)\right] + F_p(n - 1 - p) - F_p(n - k) \\ &\vdots \end{aligned}$$

$$= \left[F_p(n-k) + F_p(n-k-p)\right] + F_p(n-k-p+1) - F_p(n-k-p+2) + \dots + F_p(n-k-p+(k-1)) - F_p(n-k)$$

$$> F_p(n-k-p)$$

$$> m$$

which yields a contradiction.

case 2: Now suppose $N = F_p(n) + m$ for some n and $0 < m < F_p(n-k)$. If we also have $N = F_p(n-k) + \overline{m}$ with $k \geqslant 1$ and $\overline{m} < F_p(n-p-k)$, then $F_p(n) + m = F_p(n-k) + \overline{m}$, and simplifying as in case 1 yields

$$F_p(n-k-p) + F_p(n-k-p+1) + \dots + F_p(n-k-p+(k-1)) + m = \overline{m}$$

and this implies $F_p(n-k-p) < \overline{m}$ which is a contradiction.

With this information, Moriarty is able to represent any two-digit number $N < 100$ as a unique pair of integers n, m and these in turn are represented by specific terms in the corresponding Fibonacci-type sequence. Consider the following example where $p = 3$ and the Fibonacci-type sequence is 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, … .

$N = F_3(n) + m$	$= F_3(n) + F_3(n_1)$	A Representation
$6 = 5 + 1$	$= F_3(5) + F_3(1)$	0501
$7 = 7 + 0$	$= F_3(6)$	06
$12 = 10 + 2$	$= F_3(7) + F_3(2)$	0702
$18 = 14 + 4$	$= F_3(8) + F_3(4)$	0804
$36 = 26 + 10$	$= F_3(10) + F_3(7)$	1007

Note that commutativity would allow the representation to be just as easily written 0105, 06, 0207, 0408, or 0710. Consider, though, what happens when $N = 25$, for then $25 = 19 + 6 = F_3(9) + m$, but $m = 6$ is not one of the terms in the $F_3(n)$ sequence. But we can write $6 = 5 + 1 = F_3(5) + F_3(1)$, which means $25 = F_3(9) + F_3(5) + F_3(1)$, and its representation is then 090501 (or 090105 or 010509).

Armed with this representation, Moriarty pulls out his favourite book (a second, but private key) and proceeds to extract the necessary characters from the book, noting their page number, line number, and character position (we count blank spaces and punctuation) in the line. It's possible that Moriarty could have chosen *Alice in Wonderland* as his favourite book because he personally knew Charles Dodgson (who, incidentally, had been Mycroft's mathematics tutor at Christ Church college), though he didn't think much of Dodgson as a mathematician [2]. Selecting letters from *Alice* might have been a means to throw Holmes off his tail. Finally, then, Moriarty has decided Holmes is too much of a threat, and wishes to instruct his number one henchman, Colonel Sebastian Moran, to dispose of Holmes.

He sends Moran (who possesses a copy of the book) the message

KILL HOLMES

by constructing the code taken from pages 12, 25 in *Alice* as follows.

Page	Line	Text
12	06	**H**mmm, what day is **i**t?
	07	It's Tu**e**sday
	08	Tsk, ts**k**, two days wrong! I to**l**d you not …
25	12	**L**ook out, fi**v**e! You're **s**plashing the paint
	13	al**l** over **m**e!

The ten letters chosen are indicated in bold. Moriarty would therefore have constructed the following table of values:

	Page	Line	Character	Representation
K	12	8	8	0702 0601 0601
I	12	6	16	0702 0501 0802
L	12	8	31	0702 0601 1005
L	25	12	1	090501 0702 01
H	12	6	1	0702 0501 01
O	12	8	22	0702 0601 0903
L	25	13	3	090501 0703 03
M	25	13	10	090501 0703 07
E	12	7	8	0702 06 0601
S	25	12	24	090501 0702 0905

To make matters even more complicated, Moriarty would have combined the representations for the L, M in ‘Holmes’ since their characters were taken from the same line on the same page, and he would have written that information as

090501 0703 03 07.

Finally, the message, in code, that Moriarty sends to Moran is

0702 0601 0601
0702 0501 0802
0702 0601 1005
090501 0702 01
0702 0501 01
0702 0601 0903
090501 0703 03 07
0702 06 0601
090501 0702 0905

although, as mentioned earlier, he may wish to commute some of the digits and, perhaps, rewrite the first line, say, as 0207 0106 0601.

Conclusion

Much credit for the inspiration of this work goes to Goriely and Moulton for constructing such an intricate scenario. Portions of the above work differ slightly from what they had, but the hope is that it will still appeal to all Holmes lovers. It is truly unfortunate that so little of this made it to the movie screen. In the brief span of time that the blackboard appeared one was lucky to have had time to focus in on Pascal's triangle. But that's Hollywood.

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3. Sir Arthur Conan Doyle, *His last bow*, Berkley Publishing Corporation (1944) pp. 80-109.
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Notes

On Fibonacci numbers that are factorials

Introduction

The nth Fibonacci number, denoted by F_n, is defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$, with initial values $F_1 = F_2 = 1$. Let $\mathcal{F}$ denote the set of Fibonacci numbers, and let $\mathcal{S}$ be some other set of positive integers. If $\mathcal{S}$ is an infinite set, we might ask whether or not $\mathcal{F} \cap \mathcal{S}$ contains an infinitude of elements. In other words, do infinitely many elements of $\mathcal{S}$ appear in $\mathcal{F}$? Furthermore, should $\mathcal{F} \cap \mathcal{S}$ contain only finitely many elements, we might ask for a list of those elements appearing in this intersection.

Let us consider, for example, the situation in which $\mathcal{S} = \{1, 4, 9, 16, \ldots\}$, the set of non-zero square numbers. It is known in this case that $\mathcal{F} \cap \mathcal{S}$ is finite [1]. In fact, it is shown in [1] that the only squares appearing in the Fibonacci sequence are $F_1 = F_2 = 1$ and $F_{12} = 144$. By way of another example, let $\mathcal{S}$ denote the set of perfect numbers. Although it is not known whether or not there are infinitely many perfect numbers, it has in fact been proved that $\mathcal{F} \cap \mathcal{S} = \varnothing$ [2, 3].

In this article we look at the case in which $\mathcal{S}$ is the set of factorials. In Theorem 1.1 of [4] a list is given (with proof) of the only factorials that can be expressed as the sum of at most three Fibonacci numbers. Of these, the only ones for which the sum comprises just one Fibonacci number are $0! = 1! = F_1 = F_2$ and $2! = F_3$. This of course implies that F_1, F_2 and F_3 are the only Fibonacci numbers that are factorials, and the answers to our questions in this case thus follow as a corollary to Theorem 1.1 of [4]. However, this theorem is certainly not straightforward to prove, and our aim here is to provide a relatively simple proof of the fact that, for any $n \geqslant 4$, F_n is not a factorial.

Some preliminaries

In this section a number of results that will be used in the proof of the theorem are gathered together. First, we obtain the following very straightforward result.

Result 1: $F_n < (n-1)!$ for all positive integers $n \geqslant 4$.

Proof: This result is proved by induction on n. First, we have

$$F_4 = 3 < 6 = 3! = (4-1)!$$

and

$$F_5 = 5 < 24 = 4! = (5-1)!,$$

so the statement of the result is certainly true for $n = 4$ and $n = 5$.

Let us now assume that $F_n < (n-1)!$ and $F_{n+1} < n!$ for some $n \geqslant 4$. Then, by way of the inductive hypothesis,

$$F_n + F_{n-1} < (n - 1)! + n!$$

and hence

$$F_{n+2} < (n + 1)(n - 1)! .$$

Since, for $n > 1$, it is the case that

$$(n + 1)(n - 1)! < n(n + 1)(n - 1)! = (n + 1)! ,$$

we see that $F_{n+2} < (n + 1)!$, thereby completing the proof.

We shall also make use of the following result, which appears in [5], noting that an alternative derivation of this (stated in a slightly different form) is given in [6].

Result 2: Let p denote any prime number such that $p \neq 5$. If $p \equiv \pm 1 \pmod 5$ then $p \mid F_{p-1}$, while if $p \equiv \pm 2 \pmod 5$ then $p \mid F_{p+1}$.

For example, $11 \equiv 1 \pmod 5$ and $F_{11-1} = F_{10} = 55 = 5 \times 11$, while $13 \equiv -2 \pmod 5$ and $F_{13+1} = F_{14} = 377 = 29 \times 13$.

Finally, we will have cause to utilise Carmichael's theorem on Fibonacci numbers [7, 8, 9]. This provides us with useful information concerning the prime factors of Fibonacci numbers, and may be stated as follows.

Result 3: Let n be a positive integer such that $n \notin \{1,2,6,12\}$. Then F_n has at least one prime factor that does not divide any earlier Fibonacci number.

By way of an illustration of this result, let us consider $F_{16} = 987 = 3 \times 7 \times 47$. Examining the prime factors in turn, we note that 3 is a factor of F_4 and 7 is a factor of $F_8 = 21$, but 47 does not appear as a factor of F_n for any $n \leqslant 15$.

Proof of the theorem

We are now in a position to provide a relatively simple proof of our main result.

Theorem 1: F_n is not a factorial for any $n \geqslant 4$.

Proof: This will be proved by way of a contradiction. To this end, suppose that $F_n = k!$ for some $n \geqslant 4$ and $k \in \mathbb{N}$. Result 1 tells us in this case that $(n - 1)! > F_n$, and hence $(n - 1)! > k!$. This in turn implies that $n - 1 > k$.

Let p_j denote the jth prime number (so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on). It is clear that there exist two consecutive prime numbers p_m and p_{m+1}, say, such that $p_m \leqslant k < p_{m+1}$. The set $P(m) = \{p_1, p_2, \ldots , p_m\}$ thus comprises all the prime numbers less than or equal to k. From this it follows that the set of prime factors of $k!$, and hence, by assumption, of F_n, corresponds precisely to $P(m)$. By the Fundamental Theorem of Arithmetic [10], we are therefore able to express F_n as:

$$F_n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$$

for some m-tuple $(a_1, a_2, \ldots, a_m)$ of positive integers.

Since $n - 1 > k$ and $k \geqslant p_m$, we have $n > p_m + 1$. It follows from this that $n > p_i + 1 > p_i - 1$ for all positive integers i such that $1 \leqslant i \leqslant m$. Thus, when $n \geqslant 4$, it is the case that $F_n > F_{p_i + 1} > F_{p_i - 1}$ for all $1 \leqslant i \leqslant m$.

Next, noting that neither of $F_4 = 3$ or $F_5 = 5$ are factorials, it must be the case that $n \geqslant 6$. From Result 2 we know that, so long as $i \neq 3$, either $p_i \mid F_{p_i - 1}$ or $p_i \mid F_{p_i + 1}$, while for the special case $i = 3$ we have $5 \mid F_5$. Thus, by virtue of the fact that $F_n > F_{p_i + 1} > F_{p_i - 1}$ and $F_n > 5$, every prime factor of F_n divides an earlier Fibonacci number. On the other hand, from Result 3 (Carmichael's theorem), we know that, apart from F_1, F_2, F_6 and F_{12}, every Fibonacci number has at least one prime factor that does not divide any of the previous Fibonacci numbers. Now, neither of $F_6 = 8$ or $F_{12} = 144$ are factorials, so it must be the case that F_n has at least one prime factor that does not divide any of the earlier Fibonacci numbers. This provides us with our contradiction, thereby completing the proof of the theorem.

It follows from Theorem 1 that, in order to ascertain which Fibonacci numbers are factorials, it remains simply to consider F_1, F_2 and F_3. As mentioned in the introduction, each of these is in fact a factorial.

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Problem 2.

The point P lies inside an equilateral triangle ABC with $\angle APB = 162°$ and $\angle APC = 114°$. The line AP meets BC at D. Determine the size of $\angle ADC$.

See the solution to the problem at journals.cambridge.org/trial_MAG

The Steiner-Lehmus angle-bisector theorem

JOHN CONWAY and ALEX RYBA

1. *Introduction*

In 1840 C. L. Lehmus sent the following problem to Charles Sturm: 'If two angle bisectors of a triangle have equal length, is the triangle necessarily isosceles?'' The answer is 'yes', and indeed we have the *reverse-comparison theorem*: Of two unequal angles, the larger has the shorter bisector (see [1, 2]).

Sturm passed the problem on to other mathematicians, in particular to the great Swiss geometer Jakob Steiner, who provided a proof. In this paper we give several proofs and discuss the old query: 'Is there a direct proof?' before suggesting that this is no longer the right question to ask.

We go on to discuss *all* cases when an angle bisector (internal or external) of some angle is equal to one of another.

2. *The schizoid scissors − an indirect proof*

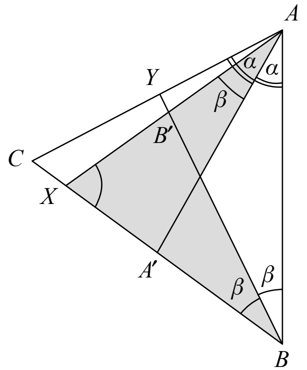


FIGURE 1: The Schizoid Scissors

The following proof is simplified from one in Coxeter and Greitzer's *Geometry revisited* [1]. We find our vivid title helps us to recall both the construction and the proof.

Suppose that one of the bisected angles A and B of triangle ABC, say the one above, 2α at A, is strictly larger than the one below, 2β at B, as in Figure 1. Then we can cut off a proper part of size β from the angle bisector AA' towards the side AC. This yields the shaded 'scissors' of the figure, whose cutting edges are the equal angle bisectors. Then the blades above and below, namely the triangles $AA'X$ and $BB'X$ share an angle γ at X and have another angle β at A or B and so are similar. We easily reach two opposite conclusions!

On the one hand, the blade above is clearly smaller that the one below since AX is opposite the smaller angle 2β and BX opposite the larger one $\alpha + \beta$ in the triangle ABX that they span. (Note that this proves $AA' < BB' < BY$ giving the reverse-comparison theorem.)

On the other hand, the blade above is larger than the one below because $AA' > BB'$, the former being a complete bisector and the latter only a proper part of one. (Note that this is the first time we have used the hypothesis that the bisectors are equal.)

This contradiction shows the bisected angles cannot be different, and so proves the theorem. However, this synthetic proof is blatantly indirect. Before discussing the directness question, we give a simple algebraic proof.

3. *An algebraic proof*

Let the lengths of the three bisectors be y_a, y_b, y_c. Then it is not too hard to see that

$$y_a^2 = bc\left\{1 - \left(\frac{a}{b + c}\right)^2\right\}$$

and from this some tedious algebra tells us that

$$(a + c)^2 (b + c)^2 \left(y_a^2 - y_b^2\right) = c(b - a)(a + b + c)\left\{(a + b + c)\left(c^2 + ab\right) + 2abc\right\}.$$

In this the factor

$$(a + b + c)\left(c^2 + ab\right) + 2abc$$

cannot vanish, proving the theorem. Also if $b > a$, then $y_a > y_b$, proving the Comparison Theorem.

Coxeter and Greitzer mention the algebraic proof and say that 'Several allegedly direct proofs have been proposed, but each of them is really an indirect proof in disguise.' It is clear from these words that they regard this algebraic proof as indirect. We now restrict ourselves to the question of whether there can be a direct proof. First we show that:

4. *There cannot be a direct proof …*

We define a process called *extraversion* ('turning inside out') of a triangle. Extraversion is a smooth process that transforms a triangle into its mirror image as in Figure 2, in which we have taken the edge AB which joins the two bisected angles (the *joining edge*) as base.

We start by moving A and B towards each other as hinted at by the bold arrows, then they pass through each other and continue until they form the reflected triangle. The numbers a and b smoothly vary but return to their initial positive values since they never pass through 0. However, c decreases uniformly, passing through zero and finishing at $-c$. In a similar way we can find what happens to the angles. When c passes through 0 so does C, and ends at $-C$, while A and B become their supplements.

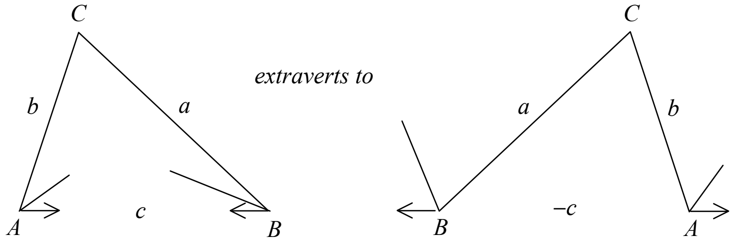


FIGURE 2: Extraverting the joining edge

Summary: c-extraversion replaces:

$$a,\ b,\ c \text{ by } a,\ b,\ -c \text{ and } A,\ B,\ C \text{ by } \pi - A,\ \pi - B,\ -C$$

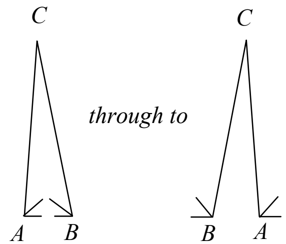


FIGURE 3: Snapshot as B passes through A so C passes through 0

Figure 3 is a snapshot of what happens as c passes from being small and positive to being small and negative. The internal bisectors at the ends A and B of the joining edge pass smoothly to external bisectors, while the bisector at C stays internal.

The direct proofs of various theorems about angle bisectors extravert to corresponding proofs of similar theorems in which some internal bisectors have been swapped with external ones. For instance, the proof that three internal bisectors meet (at an incentre) becomes a proof that one internal and two external bisectors meet (at an excentre). However we shall see later that no proof of the Steiner-Lehmus theorem can survive all such extraversions*. Under b-extraversion, our formula for $y_a^2 - y_b^2$ becomes the following:

$$(a+c)^2(c-b)^2\left(x_a^2 - y_b^2\right) = -c(a+b)(a-b+c)\left\{(a-b+c)\left(c^2-ab\right) - 2abc\right\}$$

where x_a is the length of the external bisector segment for the angle at A. However, this does not prove that $a + b = 0$; now the sign of b has

* J.H.C. confesses to having made stronger assertions that now seem unjustified.

changed we can have:

$$(a - b + c)\left(c^2 - ab\right) - 2abc = 0.$$

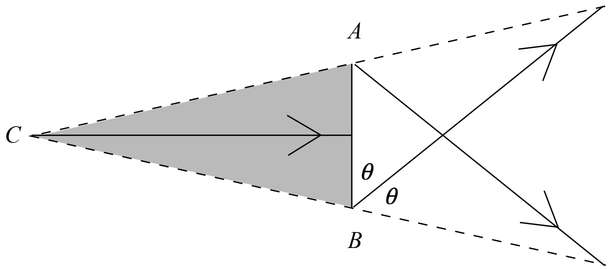


FIGURE 4: The inequilateral triangle

Indeed, the slanting external bisectors of the triangle in Figure 4 with sides 1, 1 and $-2\cos 2\theta$ are $\dfrac{-2\cos 2\theta}{\sin 3\theta}$ times as long as the vertical one. So if θ is the acute angle satisfying $\sin 3\theta + 2\cos 2\theta = 0$, namely $\sin^{-1}\dfrac{\sqrt{17} - 1}{4}$ $\approx 51.332°$, it has three bisectors (one internal and two external) of equal length, and so if Steiner-Lehmus survived extraversion, it would be equilateral. However, clearly it isn't − we call it the *inequilateral triangle* (it has angles of 77.336…, 77.336… and 25.328… degrees).

5. … *or can there*?

However, some proofs that don't survive extraversion have been considered direct. We have already remarked for instance that the algebraic proof might be considered direct. Here we consider some other plausibly direct proofs.

The schizoid scissors proof shows that both

$$\textit{above} < \textit{below} \text{ and } \textit{above} > \textit{below},$$

where *above* and *below* are any two corresponding edges of the scissor blades. Everybody will agree that use of this blatant contradiction makes the proof indirect.

But it is surely a positive statement that for any two lengths *above* and *below* we have either

$$\textit{above} \leqslant \textit{below} \text{ or } \textit{above} \geqslant \textit{below}.$$

Now if in the scissors proof we replace ‘<’ and ‘>’ by ‘⩽’ and ‘⩾’ and make a similar replacement of ‘proper part’ by ‘part or whole’, this modifies the proof to show that both

$$\textit{above} \leqslant \textit{below} \text{ and } \textit{above} \geqslant \textit{below},$$

which is no longer a contradiction, but a seemingly direct proof that

$$\textit{above} = \textit{below}.$$

6. *The direct proof that was there all along*

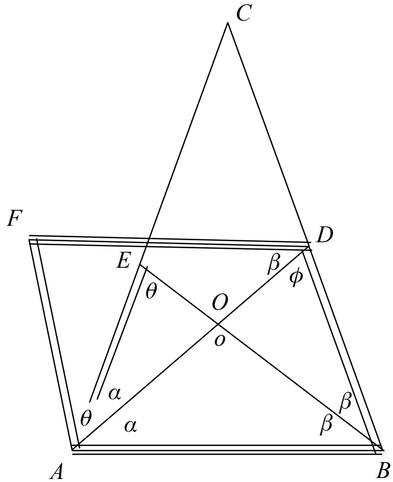


FIGURE 5: Hesse's construction

Just pOSSibly F. G. Hesse was one of the mathematicians that Sturm wrote to in 1840. In any case he produced the following proof by 1842. It uses the now generally forgotten fact (criterion OSS in the Appendix) that two triangles are congruent if they agree at two pairs of corresponding sides and a pair of corresponding non-included but obtuse angles.

In Figure 5 we picture Hesse's construction using some multiply-ruled lines. Letting *AD* and *BE* be the (1-ruled) equal angle bisectors in triangle *ABC*, Hesse constructs a 3, 2 and 1-ruled triangle *ADF* congruent to the similarly ruled triangle *EBA* with *B* and *F* on opposite sides of *AD*. The proof will show that the 3 and 2-ruled quadrilateral *ABDF* is a parallelogram.

We let *O* be the intersection of the two bisectors. Then the angle o at *O* of the triangle *OAB* is the supplement of $\alpha + \beta$. Now, because $2\alpha + 2\beta$ is less than 2 right angles, $\alpha + \beta$ is less than 1 right angle and so o is obtuse. Since it is the external angle of both the triangles *OAE* and *OBD* we have:

$$o = \alpha + \theta = \beta + \phi.$$

The forgotten fact now shows that *ABDF* is a parallelogram, since its (0-ruled!) diagonal *BF* divides it into two 3, 2, 0-ruled triangles that share this diagonal, and have two equal sides *AB* and *DF* and the obtuse angle o at corresponding vertices *A* and *D*.

Now by Hesse's construction *AE* equals *AF* which equals *BD* from the parallelogram so the triangles *ABE* and *BAD* are congruent, showing that $2\alpha = 2\beta$, and so *ABC* is isosceles.

We find it surprising that in Coxeter's long life (9 February, 1907 − 31 March, 2003) he does not seem to have commented on this proof, which in our opinion is the most ‘direct’ one.

7. *External S-L theorems*?

In the usual discussions of the Steiner-Lehmus theorem it is often supposed tacitly that angle bisectors are internal. If they are both external, there are three possible cases (MAX, MID, MIN), distinguished by whether C is the maximal, middle, or minimal one of the three angles (or equivalently whether c is the maximal, middle, or minimal one of the three edges), illustrated in Figure 6. (The switch between these cases is when C equals A or B because then the external B or A bisector is parallel to the opposite side.)

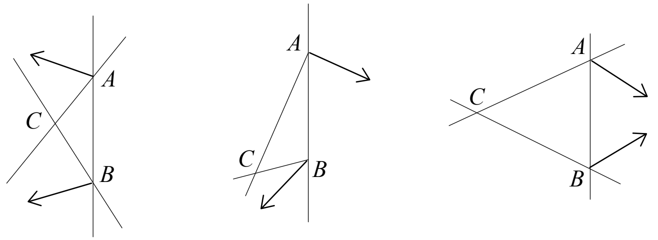


FIGURE 6: The three cases

7.1 *Case MAX: The backward external S-L theorem*

In this case, the two given bisector segments point backward (from the joining edge AB into the half-plane containing C). The theorem and scissors proof remain valid, provided that all inequalities are reversed. We suppose $\alpha > \beta$ as in Figure 7, so that we can choose X between A' and C to make the angle $A'AX$ equal β. Then for the two similar triangles AXA' and BXB', we reach two opposite conclusions (as in the internal case, but now with reversed inequalities).

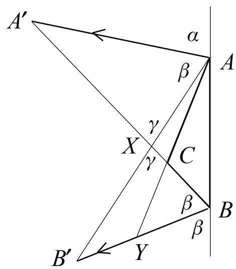
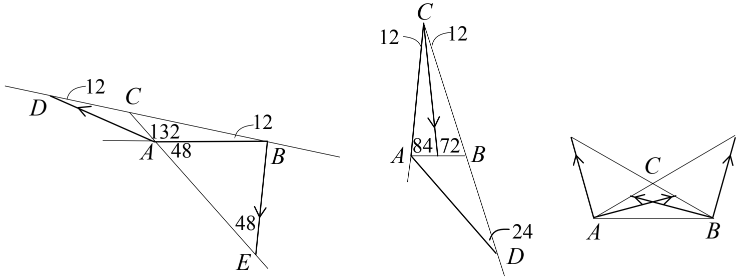


FIGURE 7: Case MAX:Backward Bisectors

In the triangle AXB, AX is opposite the larger angle $\pi - 2\beta$ while BX is opposite the smaller one $\pi - \alpha - \beta$ so AXA' is bigger. On the other hand, $AA' = BY < BB'$ so BXB' is bigger. The comparison theorem in this case is that the larger exterior angle has the longer bisector.

7.2 *Case MID: The non-theorem*

When C is between the two other angles, one external bisector is forward and the other backward. In this case, the theorem fails, a famous counterexample having been found by Oene Bottema.



(a) Bottema's integral triangle

(b) The extra integral triangle

(c) The 30-30-120 Triangle with four equal bisectors

FIGURE 8: Integral triangles

There is in fact just a 1-parameter family of counter-examples − '*Bottema's variable triangles*'. The equation that governs these is

$$(a + b - c)(c^2 + ab) - 2abc = 0,$$

in which the left-hand side is found by c-extraverting the displayed formula from Section 3. In trigonometric form this becomes

$$\sin \tfrac{5}{4}(A + B) \cos \tfrac{1}{4}(A - B) = \sin \tfrac{1}{4}(A + B) \cos \tfrac{3}{4}(A - B)$$

or equivalently

$$\sin^2\left(\frac{A - B}{4}\right) = \cos^2\left(\frac{A + B}{4}\right)\left[4 \sin^2\left(\frac{A + B}{4}\right) - 1\right].$$

The last form shows that for each value of C (or equivalently $A + B$), there is a unique value of $|A - B|$ (or equivalently the pair $\{A, B\}$). This means that if $4 \sin^2\left(\frac{1}{4}(A + B)\right) \geqslant 1$, that is $C \leqslant 60°$, there is a unique triangle for which the theorem fails. (Really this triangle is only unique up to interchange of A and B, but we shall abuse the word 'unique' in this sense whenever convenient.)

One obvious solution to the first trigonometric form of the equation has $\frac{5}{4}(A + B) = 180°$ and $\frac{3}{4}(A - B) = 90°$, yielding $A = 132°$, $B = 12°$ and $C = 36°$. This is Bottema's triangle, or more specifically *Bottema's integral triangle*, because its angles are integral in degrees. In it, the external bisectors at A and B have the same length as the joining edge, as is evident from the indicated isosceles triangles in Figure 8(a).

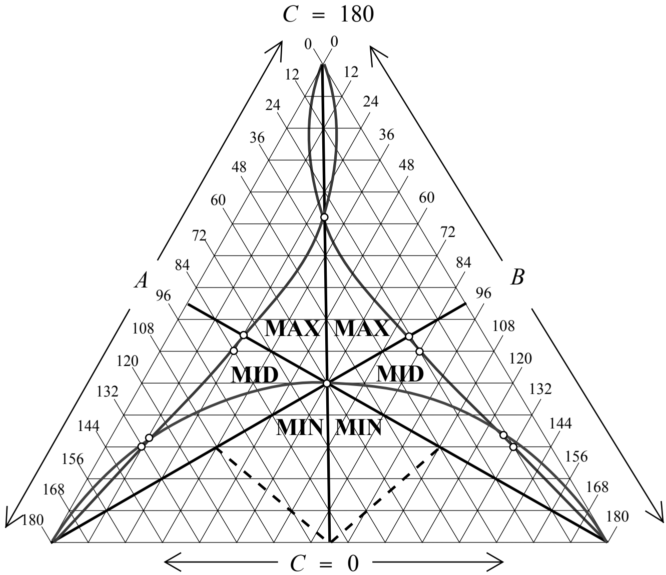


FIGURE 9: The triangle of triangles

7.3 *The triangle of triangles*

In our 'triangle of triangles', Figure 9, each point corresponds to a triple of numbers A, B and C that add to 180, and so to a shape of triangle. In the figure, A is constant on downward sloping lines, B on upward sloping ones, and C on horizontals. Isosceles triangles lie along the medians.

The approximately circular arc in the lower part of the figure contains Bottema's variable triangles and the lowest marked points on it are Bottema's integral triangle (and its A-B reflection). Above the Bottema curve the comparison theorem is direct (the larger bisector bisects the larger angle). Below it the comparison theorem is reversing (as in the internal case).

By a- and b-extraverting the equation of the Bottema curve we obtain the equations

$$(a - b + c)(c^2 - ab) - 2abc = 0 \text{ and } (a - b - c)(c^2 - ab) + 2abc = 0,$$

for the two other curves of the figure, corresponding to triangles for which an internal bisector of either A or B is equal to an external bisector of the other.

Working upwards on either of these extraverted curves, we find that after narrowly missing the Bottema integral triangle it crosses the Bottema curve at a marked point corresponding to a triangle for which the external bisector at one of A and B is equal to both bisectors at the other. (The squares of the sides of such a triangle are proportional to 1, σ^5 and σ, where $\sigma = \frac{1}{2}(\sqrt{5} - 1)$.)

It then passes through a marked point corresponding to another triangle with integral angles, the '*extra integral triangle*'. This is analogous to Bottema's, with bisected angles of 24° and 84°. Again the equal bisectors have the same length as the joining edge, see Figure 8(b).

The next marked point (on a median) is our inequilateral triangle, with three equal bisectors, and the last one (on both extraverted curves and the vertical median) corresponds to the 30, 30, 120 triangle, which has four equal bisectors (Figure 8(c)).

7.4 *Case MIN: The forward external S-L theorem*

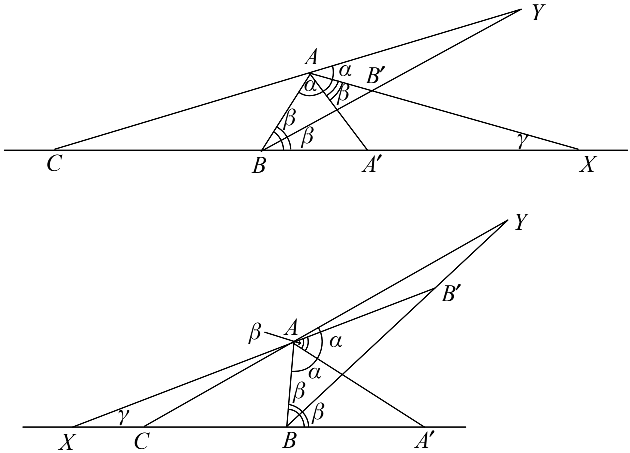


FIGURE 10: The MIN case diagram changes as X passes infinity

In this case, the usual scissor diagram takes two forms according to where the point X lies on the line BC. In the upper part of Figure 10 (where X is to the right of B) the proof for the internal case continues to work without changing a word. In particular the comparison theorem is reversing.

However, in the bottom part, X has 'passed infinity' to reappear at the left of C. Now the scissor proof argument fails because it gives $AA' > BB'$ and $BY > BB'$ which do not contradict the equality $AA' = BY$.

The boundary between the two cases is given by $2C = |A - B|$, the dashed line in the triangle of triangles. The scissor proof works only below this line, however the theorem continues to hold on and above this dashed line by the following continuity argument.

The triangle of triangles is divided into six triangular parts by its medians; inside any one of these the bisector lengths change continuously. So unless a path between two points in the same part crosses the Bottema curve, the triangles they represent must have the same comparison status (reversed or not). Since the Bottema curve lies entirely in the MID parts, this implies the comparison status is reversing everywhere in the MIN parts and direct in the MAX parts.

8. *Conclusion*

Our friend Richard Parker says that a significant mathematical assertion can be regarded as a definition of the one word you do not know in terms of the ones you do. Parker's principle suggests that when a proof of the Steiner-Lehmus theorem is described as direct, this merely tells us how the author is using the term ‘direct’. More than 170 years of discussion has taught us only that there is no agreed meaning to this term. The directness question is therefore outmoded and we should ask instead whether and where proofs involve inequalities as the extraversion argument suggests they will.

Hesse's proof does so (o is obtuse), as does the algebraic proof (a, b, c are positive), the scissors proofs (blatantly); so too do all proofs in Sherri Gardner's recent collection [3].

Our first proof makes no attempt to be direct: we call it the *strictly-schizoid proof* and, whether direct or not, the second proof certainly remains schizoid, so we call it the *still-schizoid proof*. Since it seems that every proof must involve inequalities, we are inclined to disquote George Orwell:

all proofs are inequal, but some are more inequal than others.

Appendix: Some forgotten facts

Students are warned not to be ASSes by using the ASS condition in which the angle is not included between the two sides, since this can fail as in the first part of Figure 11, which shows that there two such triangles can be different. The forgotten fact used by Hesse is that a deduction is still pOSSible when the given angle is obtuse (the OSS criterion).

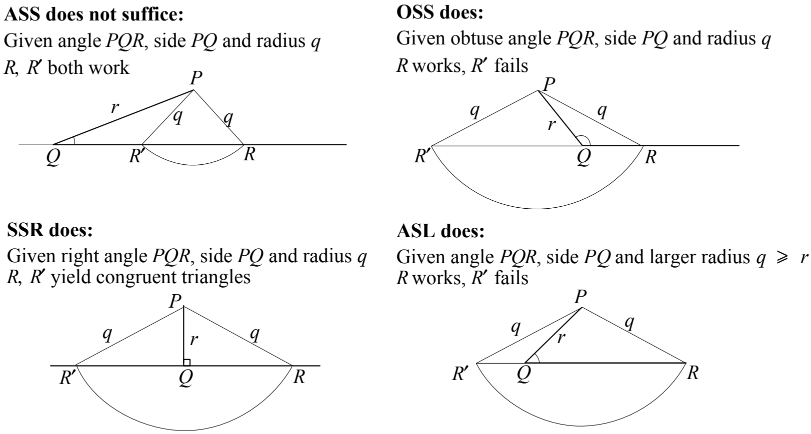


FIGURE 11: When do two sides and a non-included angle imply congruence?

The SSR case of two sides and a non-included right angle is still taught, so we call it the ‘claSSRoom’ criterion. It is interesting that this criterion with a right angle does not specialize either the acute ASS case (being valid) or the obtuse OSS one (since it permits two triangles, but these are congruent).

Figure 11 finishes with a short and simple condition that covers all valid cases* of ASS (two sides and a non-included angle), see also [4]. This is the ASL criterion, standing for 'Angle, Side, Longer (or equal) side', meaning that the given angle is opposite the longer of the two sides. Here 'longer' is to be interpreted so as to include equality. Our name for this is the 'ULTRASLICK' criterion, in which the middle-sized 'I' hints at the inclusive interpretation of 'longer'.

To include SAS along with these, we should use the ABLE condition − that (the given) Angle Belongs to a Longest Edge (of the two given ones) − note the inclusive sense. This covers *absolutely all* cases in which one is ABLE to deduce congruence from two sides and an angle!

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* Like the referee we hope that ASS can be brought back from 'the outer darkness with weeping and gnashing of teeth' to which it is usually relegated.

Combinatorics: ancient and modern, edited by Robin Wilson and John J. Watkins. Pp. 381. £55. hardcover. 2013, ISBN 978-0-19-965659-2 (Oxford University Press).

The history of mathematics is now an established and highly-regarded discipline and has prompted scholarly research into most areas of the subject, but combinatorics has, for some reason, been rather ignored. This book aims to redress the balance and presents, for the first time, a survey of the history of the field which is accessible to the general reader. Individual chapters have been contributed by a group of sixteen experts which include Eberhard Knobloch, AWF Edwards, Robin Wilson, Norman Biggs, Keith Lloyd and Ian Anderson. In addition, the distinguished computer scientist Donald Knuth presents an introductory overview of two thousand years of combinatorial investigation from ancient China to the present day.

Accordingly, the first half of the book is organised on historical principles, with sections on Indian, Chinese, Islamic and Jewish combinatorics and also the development of the subject in Europe from the Renaissance to the 17th century. It soon becomes evident that, during most of this time, combinatorics was not regarded as an area of study in its own right but was subsumed under arithmetic or algebra. Indeed, much earlier work was tied to particular problems such as enumerating the hexagrams in the I Ching, counting metrical patterns in Sanskrit prosody and classical Greek poetry and tabulating the possible combinations of theological concepts in the work of Ramon Llull, the Catalan poet and mystic. The work of the French friar Marin Mersenne led to consideration of partitions and to rules for calculating the number of permutations, arrangements and combinations of musical notes, both with and without repetition. Magic squares were, of course, well-known to artists like Durer and the ‘arithmetical triangle’, nowadays attributed to Pascal, had appeared long before his time in manuscripts from ancient India, China and Persia.

The subject begins to take a recognisable form in the 18th century as a result of the work of the Bernoullis, de Montmort, de Moivre and Stirling. Much of this was prompted by the investigation of games of chance, and there is a famous exchange of letters between Fermat and Pascal which can be said to initiate the modern study of probability. In addition, James Stirling studied infinite series and Leonhard Euler related the Königsberg bridge puzzle to a ‘geometry of position’ which would eventually become part of graph theory. Accordingly the second half of the volume is devoted to ‘modern combinatorics’ and is divided along disciplinary lines, with contributions on early and more recent graph theory, partitions, block designs, latin squares, enumeration and combinatorial set theory. A problem faced by any such account is that of striking a balance between exposition − explaining to a lay reader what the principal concepts and results are and what methods were employed in their investigation − and the strictly historical question of placing these developments in the context of the history of the subject as a whole and deciding who discovered what when. Another major issue is that of notation and terminology: how much should old research be viewed through the prism of modern usage? The authors in this volume cope with these issues in different ways, but as a whole the narrative reads lucidly and there is an extensive bibliography at the end of each chapter so that interested readers can pursue particular topics as they wish.

Inevitably we encounter many cases of mistaken attribution. Both Hamiltonian cycles and Steiner triple systems, for example, were studied by Thomas Kirkman before anyone else, but he did not receive the credit for either. Combinatorial questions also have a remarkable longevity, and keep on ‘coming back for more’ in terms of increasing refinement. The four colour problem is one whose roots go back to 1852 and which stimulated research from many ‘amateur’ mathematicians such as Alfred Kempe, whose wonderful flawed proof contained features used in the eventual (and controversial) solution by Appel and Haken in 1977. The technique of generating functions is another perennial, used with great ingenuity by Euler in the study of partitions and by Nicholson for enumeration, but still inspiring new research by Hardy and Ramanujan in 1918. Much progress in combinatorics was stimulated by the design of statistical experiments, particularly in agriculture, but the ideas employed first appeared in puzzles such as Kirkman's schoolgirls problem which appeared ‘versified by a lady’ in the *Educational Times* of 1870. Latin squares also have a very long history; indeed, the cover of the book is an illustration from the *Scientific American* of two orthogonal squares of order 10, which were also used, we are told, as the design for a needlepoint rug. Maybe the oddest appearance of any combinatorial object is the knight's tour through this array which is subject of Georges Perec's novel *La Vie Mode d'Emploi*. In the twentieth century there was renewed interest in finite sets, and particularly matters relating to their intersections, unions and orderings. One of the offshoots of this was the work of Frank Ramsey, who died tragically young in 1930, but whose seminal investigation of non-chaotic behaviour in random structures supplied a spur to further research to combinatorialists such as Paul Erdös.

A final chapter by Peter Cameron gives a quick overview of recent developments in combinatorics and reflects on how it is likely to develop. The subject, now at the forefront of much significant research, has links with group theory, mathematical logic, computer science, number theory, coding theory and algebraic geometry, as well as practical implications for the Human Genome Project and the unification of physics. He ends with a proposal that the basic currency of the universe may not be space and time but information measured in bits, suggesting that the ‘theory of everything’ may turn out to be combinatorial.

The book is beautifully illustrated with portraits of the leading contributors, reproductions of frontispieces from their books and a plethora of diagrams, both from original sources and specially drawn for the text. This fascinating survey of the history of an important area in mathematical thought deserves a place in every respectable library.

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