PRESIDENTIAL ESSAY NO. 12: MARCH 2022

We Shall See: Great Mathematical Figures from Great Mathematical Figures

By Chris Pritchard (chrispritchard2@aol.com)

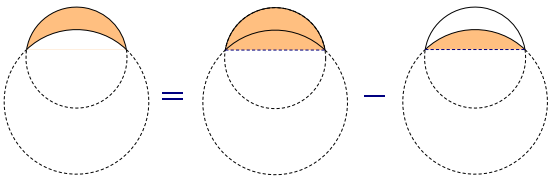
With my series of Presidential Essays coming to a close, it seems appropriate to finish with a round-up of some of my favourite mathematical diagrams. After all, the overarching theme has been the visual in mathematics and mathematics learning and teaching and this will also be the theme of my Presidential Address on 14 April.

So what wonderful mathematical figures have been produced by our great mathematical figures of the past (and present) and why are they so wonderful?

Section A
Four diagrams from the ancient world

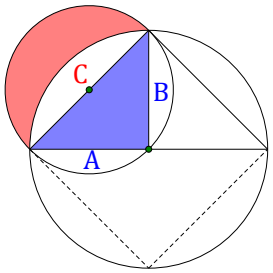
1. Hipparchus' quadrature of the lune

Hipparchus of Chios was the first mathematician to find the area of a shape bounded by curves. The shape involved is called a lune because of its likeness to a crescent moon. We may think of it as the difference between the segments of intersecting circles.



Hipparchus' result was included in his *Elements*, compiled about 440 BC, and possibly a model for Euclid's collection of the same name. Unfortunately, this work is lost but we know that the result was restated by Eudemus of Rhodes in his *History of Geometry* the following century. This history did not survive antiquity either but it had been read in its day by a number of Greek mathematicians and philosophers. The diagram below, therefore is very much a reconstruction and, to me at least, it is not clear whether the deployment of Pythagoras' Theorem in the more general version that has similar figures on the three sides of a right-angled triangle, was available to Hipparchus. So if it's a bit of a fudge, then at least it's a fudge that leaves a sweet taste in the mouth.

The diagram shows a particular lune, the smaller circle being drawn on one side of a square inscribed in the larger circle.

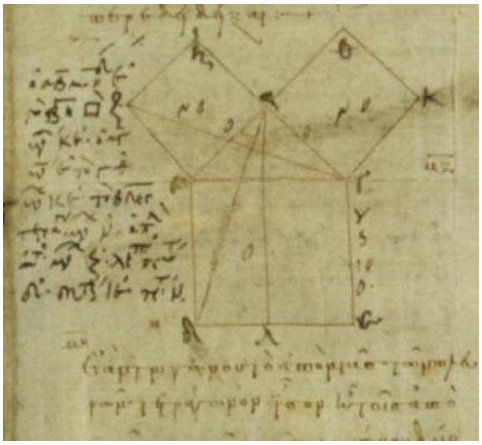


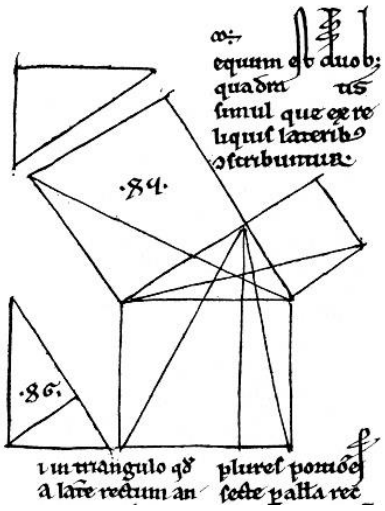
The segments labelled A, B and C are similar, so by Pythagoras' Theorem, segment C has the same area as segments A and B taken together. But segment C and the lune make a semicircle, and segments A and B, plus the triangle, make a semicircle of the same size. So the lune has the same area as the triangle. And, critically, π doesn't feature – the area of the lune is a rational multiple of the area of the square.

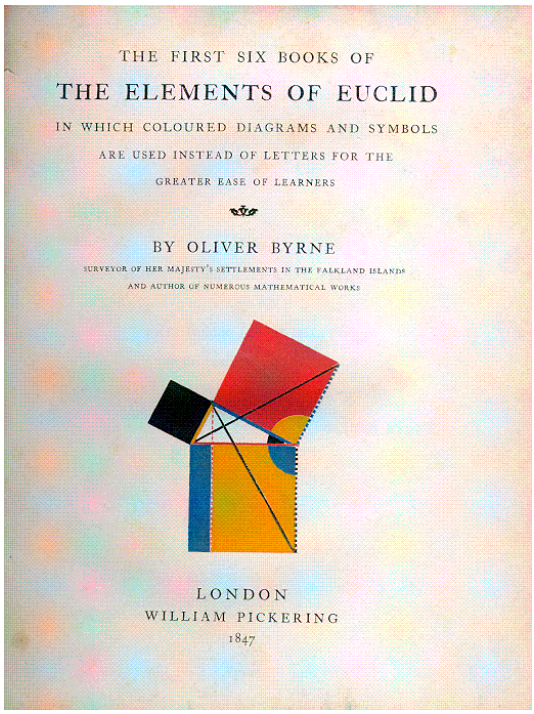
2. The Bride's Chair in Euclid's *Elements*

Somewhere around 300 BC, Euclid brought together all the known geometrical theorems (and some other results) into an ordered sequence from most elementary to most complex in his thirteen books called the *Elements*. The first book begins with self-evident truths or axioms and derives 48 propositions, culminating in Pythagoras' Theorem (Proposition 47) and its converse (Proposition 48).

The number of different proofs of Pythagoras' Theorem that have been produced over the centuries is huge and still growing. Euclid's version makes use of a diagram sometimes referred to as the Bride's Chair, after a Byzantine mistranslation of the Greek for 'winged insect' as 'figure of the bride'. Here we see it in a ninth-century Greek parchment of the *Elements*, in Medieval Latin and in an exquisitely-illustrated version published in Victorian England.





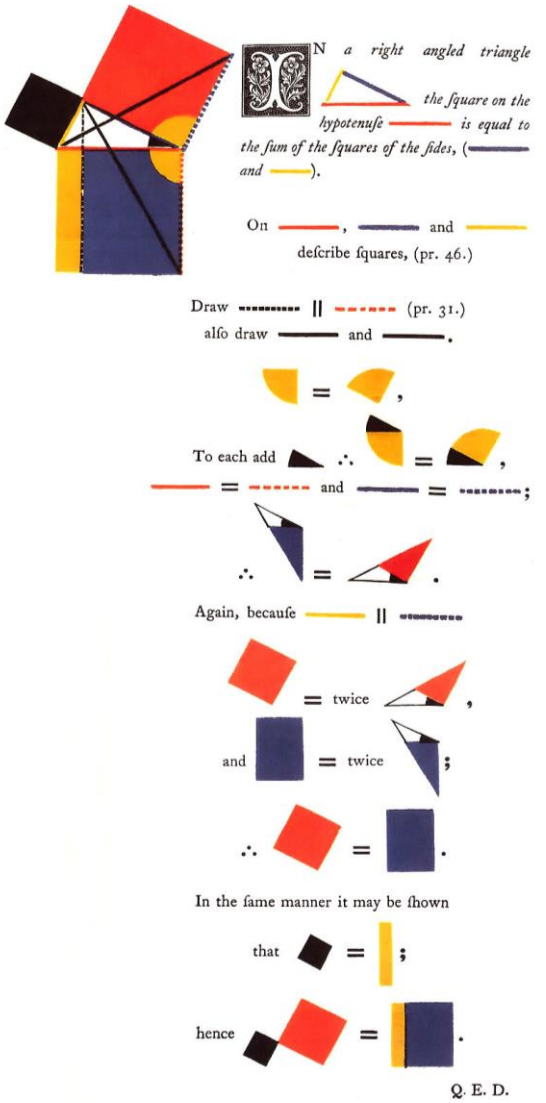


I would like to focus on the last of them, simply because of its beauty. It comes from an Irish civil engineer, Oliver Byrne (1810–1880), at one time Surveyor of Her Majesty's Settlements in the Falkland Islands whilst also a vociferous opponent of British rule in Ireland. Like William Playfair (see Essay 4), Byrne also spent time in a debtor's prison, Whitecross Street Prison, Islington, in his case, rather than the Fleet. Both prisons feature in *The Pickwick Papers*.



Byrne first experimented with the use of colour in a pamphlet of 1831, *The Teaching of Geometry by Coloured Diagrams*. It covered the first book of the *Elements* and so presumably we can date his version of the Bride's Chair to that year. The version we see here is taken from *The First Six Books of the Elements of Euclid in Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners* which was published in 1847. As you can see, the Bride's Chair appears on the title page. The Mathematical Association holds a copy of this book in its library collection housed at the University of Leicester. (Seek advice from the Association's headquarters should you wish to view it.)

At school, I learnt Euclid's proof in glorious monochrome. Here I will simply let you see how Byrne lays it out in his inimitable, colourful way. The proof makes use of congruent triangles to split the square on the hypotenuse into two rectangles each with an area the same as one of the smaller squares. See if you can follow it.



Notes

- For an account of Oliver Byrne's life, see Susan M. Hawes & Sid Kolpas, 'Oliver Byrne, the Matisse of mathematics', *Convergence* (August 2015), MAA;

  www.maa.org/press/periodicals/convergence/oliver-byrne-the-matisse-of-mathematics

- There has been a spate of recent republishing of Bryne's book.

  Taschen (the art publishers) are flagging a copy to be released in February 2022: www.amazon.co.uk/Oliver-Byrne-First-Elements-Euclid/dp/3836577380 .

  Kronecker-Wallis describe a recent book completing all 13 books of Euclid in the Byrne style:

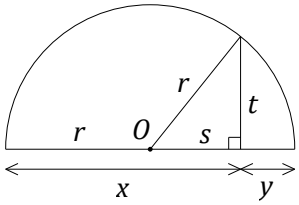
  www.kroneckerwallis.com/product/euclids-elements-completing-oliver-byrnes-work/

  There is an on-line project on Byrne at: www.c82.net/euclid/about/

3. Pappus' comparison of means

Pappus is one of the last great mathematicians of ancient Greece. His *Synagoge* or *(Mathematical) Collection* was published in eight books of which the first is lost as are parts of the other seven. Though it was written about 340 AD, the oldest extant copy, on parchment, is held in the Vatican library and is from the 10th century.

In the second section of Book III, Pappus gave the equivalent of the following figure.

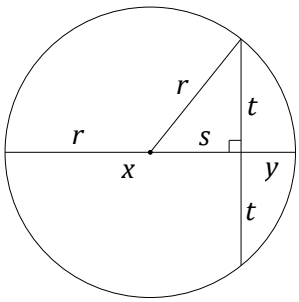


Pappus used the figure to prove why the arithmetic mean of two distinct positive numbers is greater than the geometric mean. Here those numbers are expressed as lengths, x and y. Putting them end to end, Pappus used their sum to form the diameter of a semicircle of radius r, say. If a perpendicular of length t is constructed to the circumference, a right-angled triangle is created, the base of which has length s, say. Now, in modern notation we can say that:

$$2r = x + y \quad \Rightarrow \quad r = \frac{1}{2}(x + y).$$

So r represents the arithmetic mean of x and y.

Now use reflection to duplicate the figure.



By the intersecting chords theorem, which Pappus would have known as Euclid III, 35,

$$t^2 = xy \quad \Rightarrow \quad t = \sqrt{xy}\,.$$

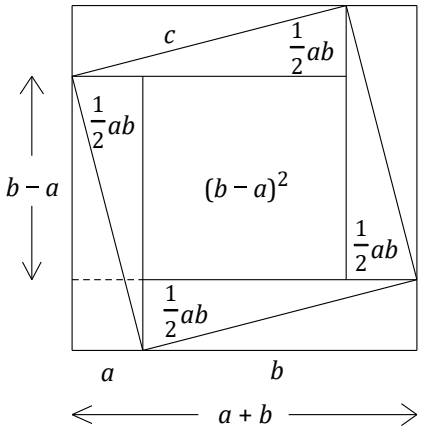
So t represents the geometric mean of x and y. But clearly, since r is the length of the hypotenuse of the right-angled triangle it is greater than t and nothing further need be said. Or perhaps it should because there's a nice spin-off for our teaching. Note that $x = r + s$ and $y = r - s$, so that

$$t^2 = xy = (r + s)(r - s) = r^2 - s^2$$

and Pythagoras' Theorem follows directly.

4. Zhào's lost diagram

In the third century AD, Zhào Shuǎng wrote a *Commentary on the Zhōubi Suànjing* (Zhou was an earlier dynasty, Suànjing means classics of arithmetic), which included an 'Illustrated commentary on the right triangle, circle and square'. The use of the word 'illustrated' suggests that the work was enhanced by figures which helped explain the 21 theorems, given in algorithmic form using specific examples. Unfortunately, the diagrams from this text have been lost, but one of them, the 'diagram on the hypotenuse' has been reconstructed by Lǐ and Dù. (See Lǐ Yan & Dù Shíràn, *Chinese Mathematics, A Concise History*, Clarendon Press, Oxford, 1987.)



By equating areas, it can, and was, used to show that

$$c^2 = (a+b)^2 - 2ab = (b-a)^2 + 2ab,$$

from which Pythagoras' Theorem follows by either of two routes. Associated diagrams (also lost) enabled Zhào to solve some quadratic equations and is similar to methods found later in India and the Middle East. Can you find other relationships in the figure?

Section B
'Pascal's Triangle' and its Predecessors

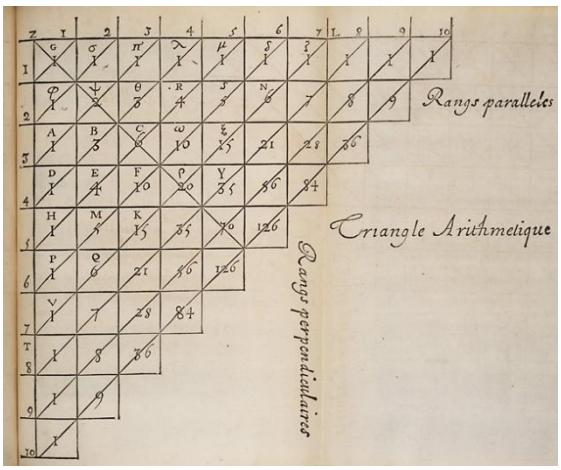
This section takes a different form from the first, a time-reversed take on the triangle of binomial coefficient we now call Pascal's triangle.

5(a) Blaise Pascal (and Pierre de Fermat)

In the second half of 1654, Blaise Pascal (1623–1662) wrote a 35-page *Treatise on the Arithmetical Triangle* in two parts, the first on the triangle itself and the second on its application to the theory of figurate numbers, to the theory of combinations, to the division of stakes problem and to finding binomial coefficients.

The section on the division of stakes problem could not have been written before July because it was in that month that Pascal and Pierre de Fermat (1601–1665) entered into a correspondence which has since been marked as the genesis of probability theory. The problem is this: if two players fail to complete a game of chance, how should the stakes be divided on the basis of the state of play at the point of interruption? Let's say that a player needs six points to win and the game comes to an untimely conclusion when the players have five points and three points respectively, and the stakes are of a particular fixed amount. This is essentially the problem tackled unsuccessfully by Luca Pacioli in 1494, Tartaglia in 1556 and Peverone in 1558. A contemporary of the last two, Girolamo Cardano did recognise the importance of focusing on the unplayed points, the missing endgame. Having previously shared the problem with Fermat, Pascal explained to him how the solution lay in binomial coefficients. Meanwhile, Fermat had established the same results using figurate numbers and their letters past in the post.

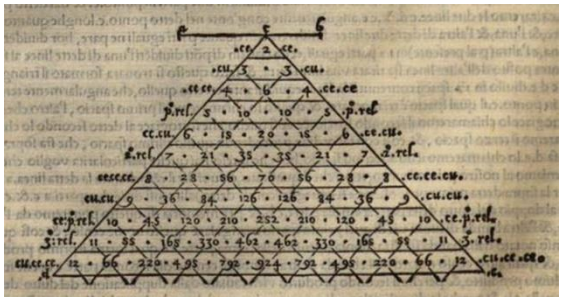
Here is the triangle as published in Pascal's treatise. We perhaps note that the orientation is somewhat different from the way we see it today.



The question arises as to whether this diagram should have Pascal's name attached to it.

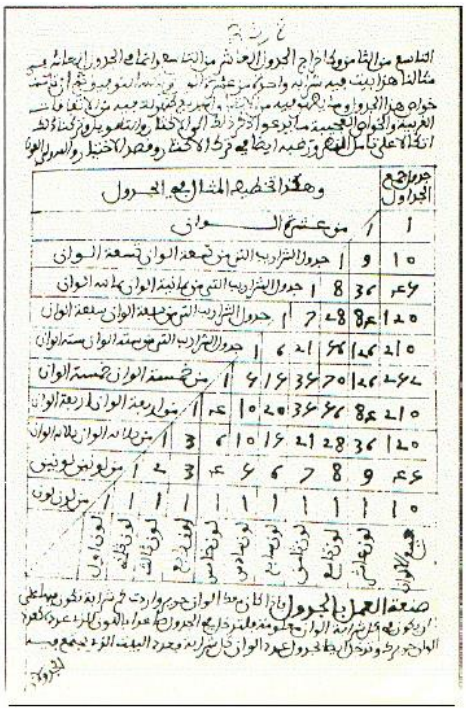
5(b) Tartaglia

To this day, the triangle is known in Italy as Tartaglia's triangle. Niccolo Fontana was dubbed Tartaglia, the stammerer, as a child, after a French soldier used his sabre to inflict a horrific facial wound whilst he sheltered in Brescia cathedral. The second part of his three-part *Treatise on Numbers and Measures* (1556) contained the triangle shown here.



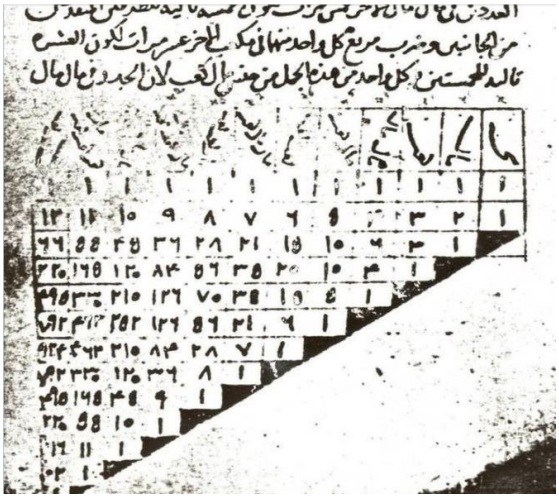
5(c) Ibn Munim

Ibn Munim, Andalusian by birth, living in what is now Marrakesh, produced three mathematical texts around 1200 of which only the last, *Fiqh al-hisab*, has come down to us. It contains the first tract solely on combinatorics and includes the following figure in the context of finding the number of ways of combining silks of different colours for which the Maghreb was famous.



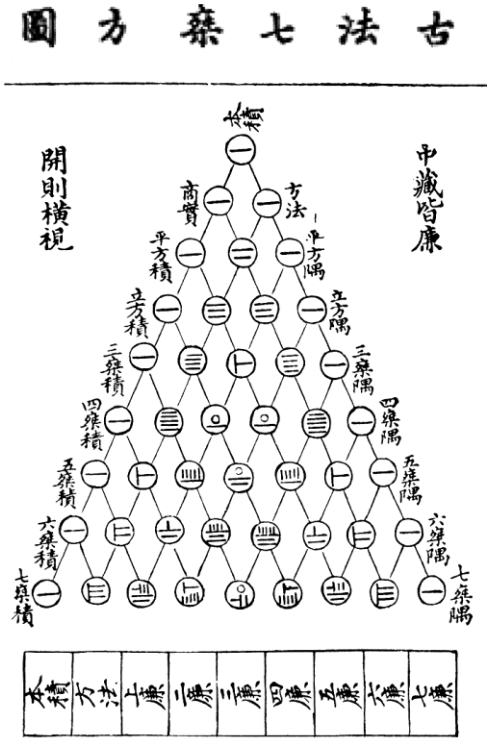
5(d) Al-Kharaji

The Persian mathematician and hydrologist, Muhammad Al-Karaji (sometimes, wrongly dubbed Al-Kharki), lived in Baghdad. At the start of the eleventh century he published a table of binomial coefficients, shown below, which was later discussed by another Baghdad-based scholar, Omar Khayyam (who gave us the words 'algebra' and 'algorithm' and, more importantly, some phenomenal mathematics to go with them).



5(e) Jia Xian

At much the same time in China, Jia Xian used the triangle to find roots. Yang Hui included an image of the design in 1261, and the diagram shown here is from 1301.

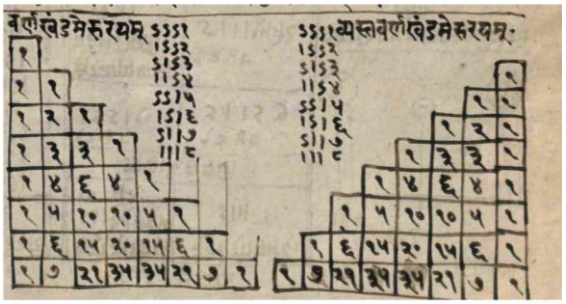


[For more details on the triangle over the centuries, see Robin Wilson & John J. Watkins (eds.), *Combinatorics: Ancient & Modern*, Oxford University Press, 2013.]

5(f) Acharya Pingala

Finally, the image below shows an algorithm known as the 'meru prastara' given by Acharya Pingala in his *Chandaḥśāstra*. We know of this from a 10th century commentary of Pingala's

work by Halayudha. The version shown here is dated 775 AD, though I wish I could say I am completely convinced by its authenticity.



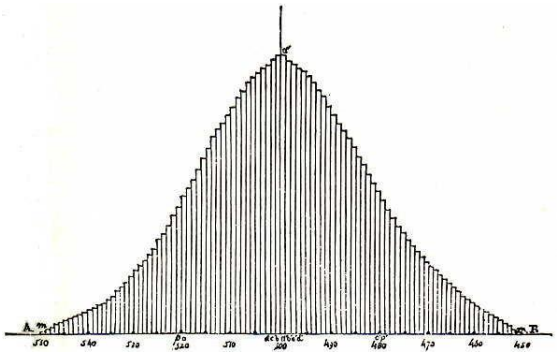
What is clear, however, is that the arithmetic triangle is extremely ancient – over 2000 years if the Pingala attribution is correct and a thousand years otherwise. This means that the attachment to it of Pascal's name or indeed Tartaglia's name is wholly unwarranted.

Section C

Some statistical diagrams

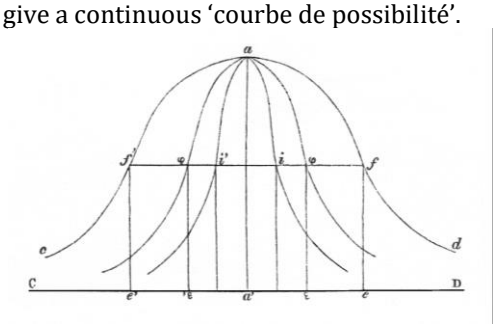
6. Quetelet on the Normal Approximation to the Binomial

In the early history of statistics there are few more influential men than Adolphe Quetelet (1796–1874), whose data sets on the chest girths of Scots militiamen and the heights of French conscripts were used to establish the 'law of frequency of error' (later the normal distribution) as a model of the variability in human physical dimensions. It would pave the way for the work of Francis Galton and others in the second half of the nineteenth century.



But Quetelet would not have reached this point without taking the earlier step of recognising the normal curve as the limiting form of the binomial. Now this in itself was not new in the mid-1840s because Abraham de Moivre had already made use of Stirling's formula to give the result without recourse to a diagram in a pamphlet of 1733, and Christian Kramp (1760–1826) had produced tables of the normal integral in 1799 in relation to his optical researches. Quetelet produced the figure above to depict for the first time the binomial distribution behaving like the normal, asymptotically, i.e. for a large number of trials.

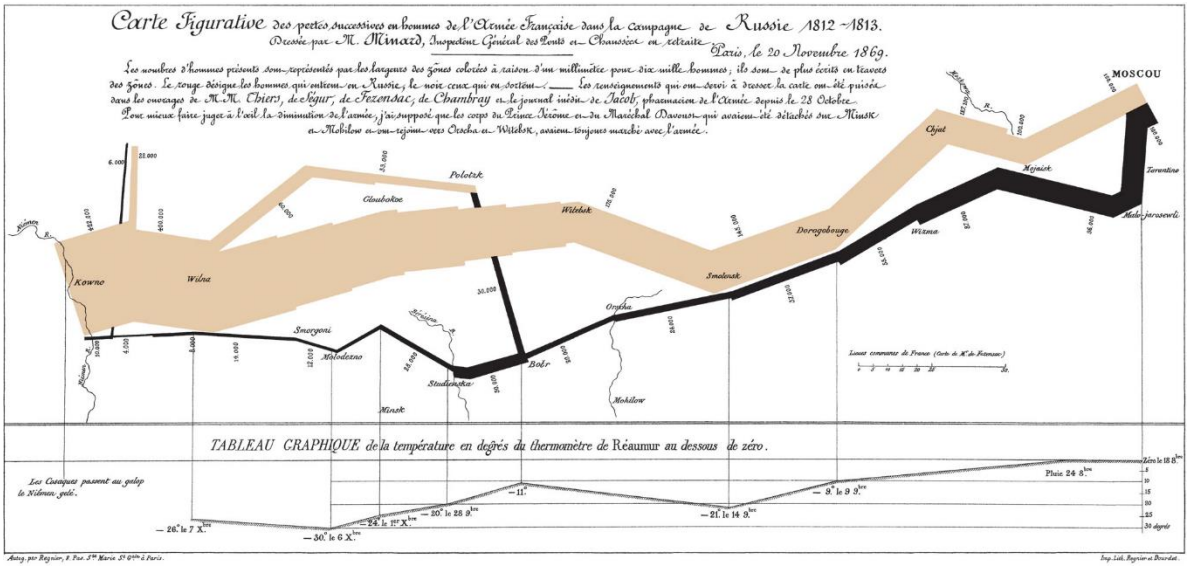
It appeared in *Lettres … sur la Théorie des Probabilités*, published in 1846, though the arguments had been rehearsed in a monograph two years earlier. Quetelet considered making 999 draws from an urn holding a vast number of white and black balls in equal proportions. The probabilities were given in a table beginning with the most likely result of 499 white and 500 black balls and finishing with 420 whites and 579 blacks. These results were displayed in a diagram of a symmetric binomial distribution, with its 'ligne brisée' (broken line), but also generalised to give a continuous 'courbe de possibilité'.



Born in Ghent, Quetelet established a significant reputation as an astronomer and founded the Brussels Observatory, but in applying the normal curve to human dimensions he also founded the science of anthropometry.

7. Minard on the strength of Napoleon's army

We met Charles Joseph Minard (1781–1870) and a number of his wonderful statistical diagrams in Essay 10 but I decided against including his most famous diagram, a time-series chart with spatial dimensions. Once described as 'seeming to defy the pen of the historian by its brutal eloquence', it shows the strength of Napoleon's army as it marched on Moscow and as it retreated. The thickness of the tan path indicates the number of men available to him, from over 400,000 at the outset of the campaign to just a quarter of that as he reached Moscow, then (if we follow the darker path) the retreat, with the number of men still in decline until only 10,000 remain as the army regains Poland. In this pictorial story of woe, from battle losses to extreme cold, hunger fatigue and death, Minard demonstrates the decimation of Napoleon's army to wonderful effect. It plots six variables, **strength** and **position** (two-dimensions) in **time** of the army and its **direction of movement** as well as **temperature,** In *The Visual Display of Quantitative Information*, Edward Tufte commented that it "may well be the best statistical graphic ever drawn".



8. Spear and Tukey on box plots

Box plots are skeletal, statistical diagrams in which prominence is given to key details, such as the quartiles, sometimes the maximum and minimum, so that the interquartile range and the range are in clear view, and a comparison of the intervals between Q_1 and Q_2 and between Q_2 and Q_3 hint at the distribution's symmetry or asymmetry.

Mary Eleanor Spear, née Hunt (1897–1986) gave us the first box plot in *Charting Statistics* (McGraw Hill, 1952). Building on the ideas of Kenneth Haemer, she referred to it as a 'range bar'. There are two examples on page 166, the image below being of the latter.

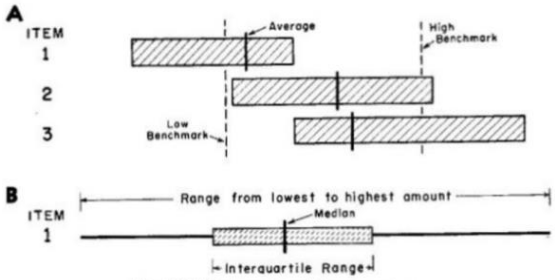
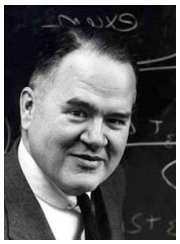


Fig. 6-24. Various uses of the range bar.

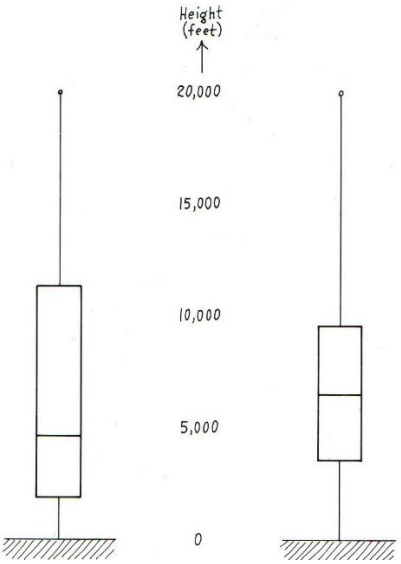
It is immediately apparent that a strength of such diagrams is their use as a quick comparison between data sets. The expanded version of this book, *Practical Charting Techniques* (McGraw Hill, 1969) began with the words, "The response to a visual presentation will determine its value."



Spear worked for a number of US governmental agencies, including the Internal Revenue Service and the Bureau of Labor Statistics in a forty year career, much of the time producing charts of economics data. Alongside, she ran her own graphics studio and taught at the American University in Washington.



John Tukey (1915–2000) was a pioneering mathematician and statistician with an eye for developments in computer technology. He gave us the words 'bit' and 'software', probably helped with the development of the U-2 spy plane (such things are not spoken about openly) and gave us a raft of techniques in statistics. In 1965 he became the inaugural director of Princeton University's statistics department. To many of us in education, he's best known for his promotion of the box-and-whisker plot in his *Exploratory Data Analysis* (Addison-Wesley, 1977). Two such plots appear first on page 40, one for the distribution of highest points in the fifty states (left) and the other for the heights of volcanoes.

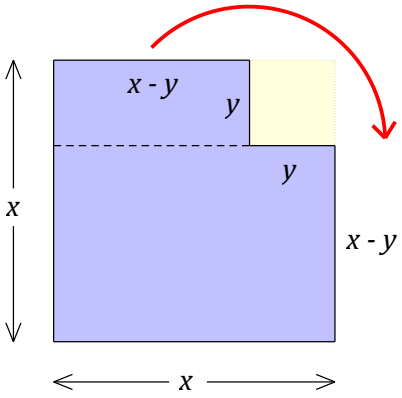


No mention is made by Tukey of Mary Eleanor Spear's primacy in the design of these statistical diagrams. In fact, he makes no mention of her at all. Her lack of a major academic position, and quite likely her gender, contrived to rob her of appropriate acclaim in her lifetime.

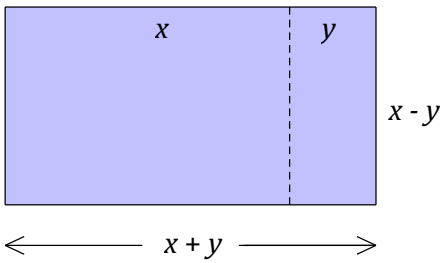
Section D
Pedagogical diagrams

9. Difference of two squares and difference of two cubes

If we cut a small square of side y from a larger square of side x, we would perhaps picture as on the left here:

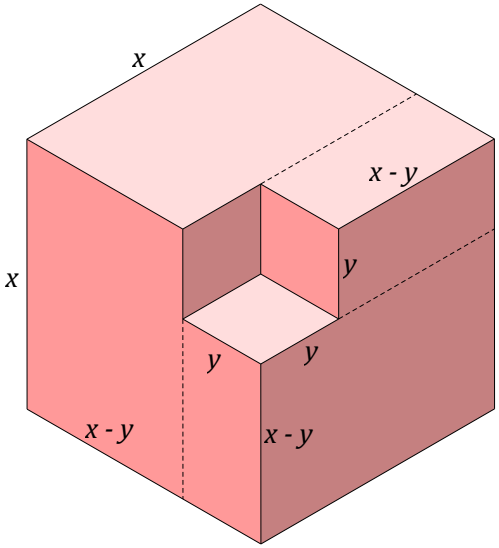


If we cut along the dotted line, we can turn the upper rectangle in the direction indicated by the red arrow. Then reattaching the two pieces we have a rectangle of base $x + y$ and height $x - y$.



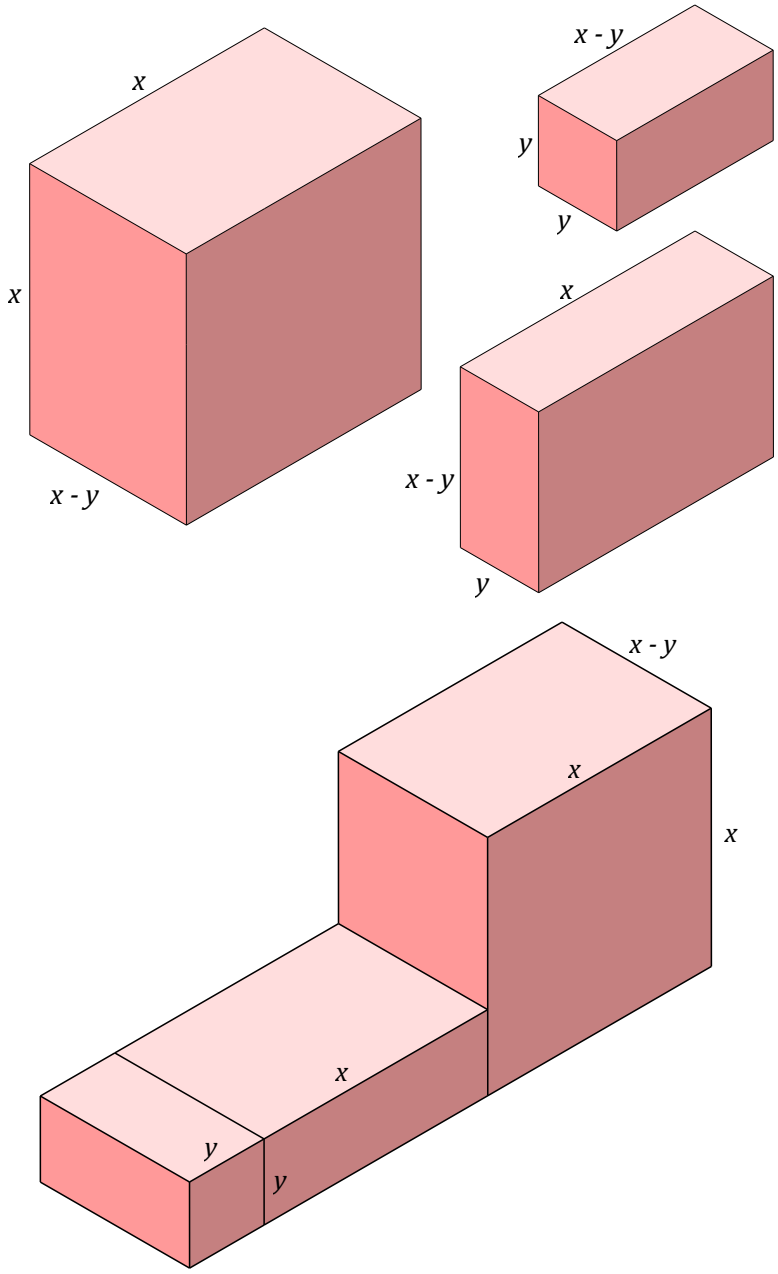
Since the areas of the initial and final figures are equal, $x^2 - y^2 = (x - y)(x + y)$. Note that the cut is made in such a way that the two pieces have a side in common (hence the convenient reattachment) and the length of that common side is one of the factors. So the dissection and rearrangement demonstration not only gives the correct difference of squares factorisation but we can see the common factor being extracted.

Exactly the same approach can be used to factorise a difference of cubes, though of course we have to step up to three-dimensional figures. Now we cut a small cube of edge y from a cube of edge x.



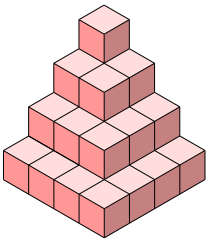
With two felicitous cuts we can produce three cuboids with a common dimension or common factor $x - y$. The pieces then be reassembled so that the common dimension is aligned to produce a prism, i.e. a shape with a uniform cross-section, which is also the second factor.

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

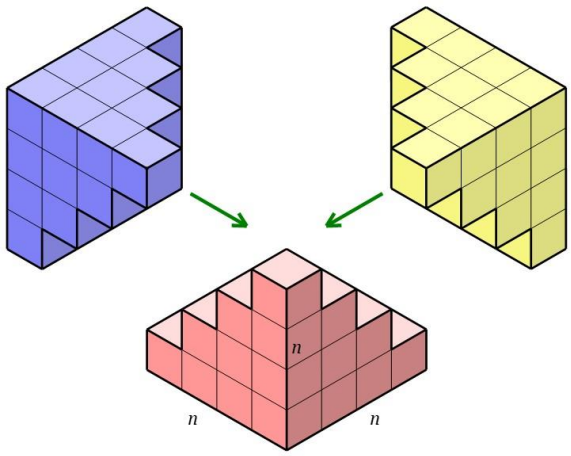


10. Sum of squares

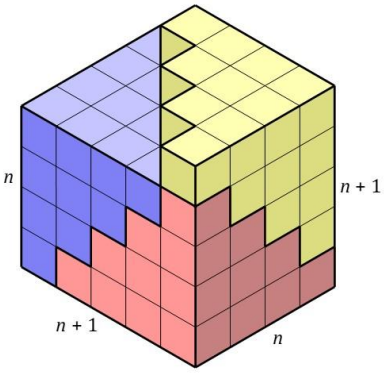
The small pyramid below is designed to show a sum of squares. Yes, it has only the first four of them, but we can imagine the pyramid with as many layers as necessary. We are imagining the full generality of the situation, i.e. we're thinking *n*, not specifically 4.



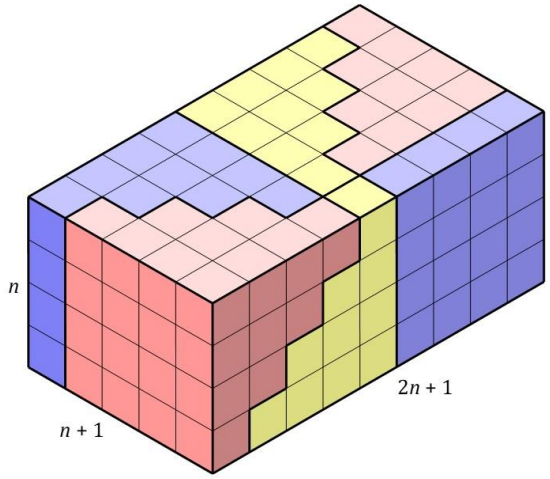
Now take three such pyramids and imagine them being fitted together in the directions of the arrows.



This produces a shape with a base n by $n + 1$ and having the first n layers complete but the $(n + 1)$th layer only half-complete.



If we put two of these compound figures together we get a cuboid of dimensions n, $n + 1$, $2n + 1$, and hence with volume $n(n + 1)(2n + 1)$.



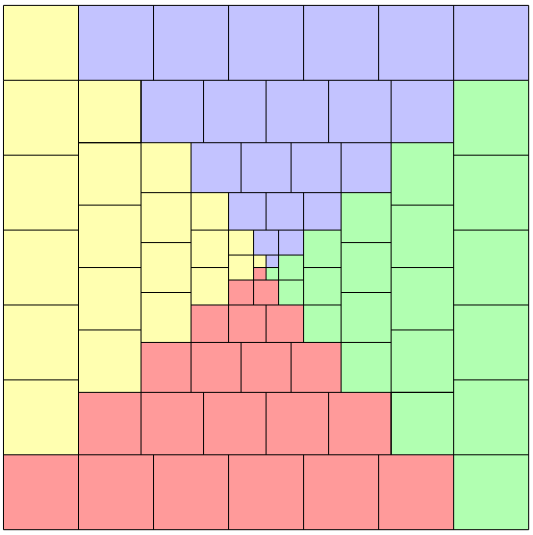
But there are six pyramids in this cuboid and so our sum of squares is represented by one-sixth of the volume of the final cuboid.

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1).$$

Figures like these were first produced by Man Keung Siu, appearing in *Mathematics Magazine* in 1984.

11. Sum of cubes

The figure below was first published by Solomon Golomb in *The Mathematical Gazette* (Vol. 49, no. 368, p. 200), though he cites Warren Lushbaugh as the inspiration. (Lushbaugh provided all the diagrams for Golomb's seminal work on polyominoes the same year.)



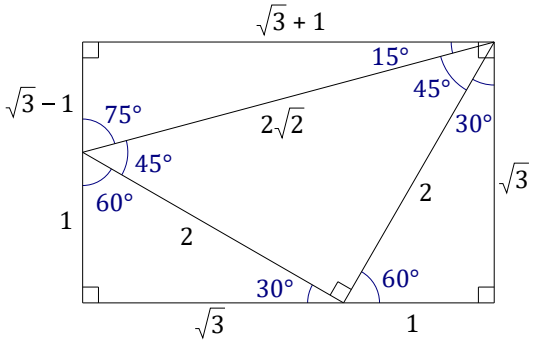
The construction is devilishly clever. Focus on the small red square with one vertex at the centre of the whole figure. Let this square have unit side length. That is, there is one square of side 1 by 1. Beneath it are two squares of side 2 by 2, three squares of side 3 by 3 … and so on. These areas give us our sum of cubes $1^3 + 2^3 + 3^3 + \cdots$.

If we take the squares in the bottom row to have side n and that there are n of them plus a green one of the same size. In other words, there are $n + 1$ squares of side n along the bottom. So, the area of the whole figure is $[n(n + 1)]^2$. But this represents four sums of cubes, so

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}[n(n + 1)]^2 = \frac{1}{4}n^2(n + 1)^2.$$

12. The Ailles rectangle

Named after a teacher in Toronto, the Ailles rectangle consists of an isosceles right triangle boxed by two 30°-60°-90° triangles and a 15°-75°-90° triangle.



Flowing from the figure immediately are

$$\sin 30° = \cos 60° = \frac{1}{2},$$

$$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \frac{1}{\sqrt{3}};\ \tan 60° = \sqrt{3},$$

$$\sin 45° = \cos 45° = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

The bonus is that we also get, simply:

$$\sin 15° = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

$$\sin 75° = \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

$$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1};\ \tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

A huge amount of useful information is stored in this simple diagram, which can be reconstructed from scratch each time it is needed by the learner.

13. Lattice multiplication

Three for the price of one here, as lattice multiplication can be used to good pedagogic effect in arithmetic, algebra and differentiation. There are numerous mathematical entities to which we could attach the adjective 'binary', by which I simply mean here a quantity with two parts. For example

- 27 $(20 + 7)$
- $x + 8$
- $x + \delta x$

If we were to multiply two such entities together then an area diagram can be helpful. So, 23×17 might be represented by

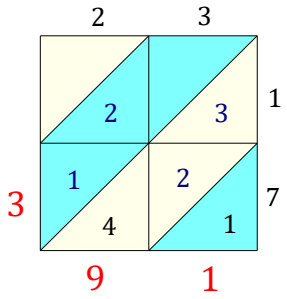
	10	10	3
10	100	100	30
7	70	70	21

or there could be only a notional appeal to area in a simpler version in which the partial sums are consolidated into four cells.

	20	3
10	200	30
7	140	21

Row totals or column totals might be added to reach the required product of 391.

In another simplification that emphasises place value, a grid is used and the totting up takes place on the diagonal. Since it arrived with us via the Maghreb and Italy, this form of multiplication takes any of the names 'shabakh' (Arabic), 'gelosia' (Italian) or 'lattice'.



Napier's bones are an aid to lattice multiplication.

Now imagine we are multiplying together two linear terms, $5x + 2$ and $3x + 8$. We can use (notional) area in the form of cells like this:

	$5x$	2
$3x$	$15x^2$	$6x$
8	$40x$	16

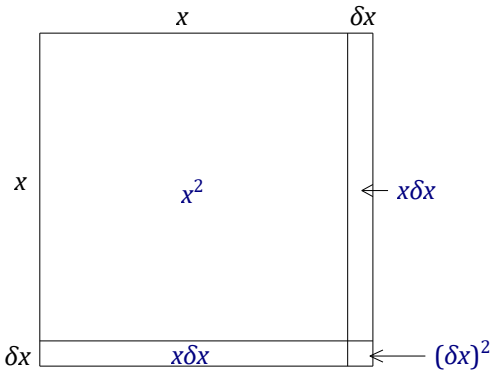
This not only gives the product by inspection as

$$15x^2 + 46x + 16$$

but emphasises the partial sums arising from the distributive law.

$$\begin{aligned}(5x + 2)(3x + 8) &= 5x(3x + 8) + 2(3x + 8)\\ &= 15x^2 + 40x + 6x + 16.\end{aligned}$$

Now let us increase x by a small amount δx so it becomes $x + \delta x$ and use a grid to depict the square of this quantity.



This is the set up given by Silvanus P. Thompson in his book *Calculus Made Easy* first published in 1910. He used it to explain that x^2 grows by an amount $2x\delta x$ plus a vanishingly small quantity

$(\delta x)^2$. If δx were as small as $1/100$, he argued, then the area of the strips would each be $1/100$ of the area of the large square but the tiny square would by $1/10000$ of the area of the large square. It would be reasonable to ignore the tiny square and consider the growth in the original square to be almost exactly the area of the strips. For a teacher, handling infinitesimals is never particularly easy. Perhaps this helps. And, essentially the same visual model is retained throughout, from elementary arithmetic to algebra, to the calculus.

Finally, a good source for diagrams for use in the classroom is the trilogy by Roger B. Nelsen under the banner *Proofs Without Words*, published by the Mathematical Association of America. (They are a little pricey but well worth it.) I learnt of the sum of squares and the sum of cubes figures from these books and much, much else.

Acknowledgement

I wish to record my thanks to Nick Lord for suggesting improvements to an earlier draft of this essay.

Presidential Essays and a Presidential Blog

As my presidential term nears its end, so do the associated essays. I will be succeeded by Dr Colin Foster of the University of Loughborough in April. He will be writing a blog and posting something every other Thursday throughout his year as President. He is intending to address issues of interest and relevance across the entire range from early years through to university – not an easy task, but he is going to do his best.

Colin hopes that the blog will stimulate conversations within the Association about important issues within mathematics teaching and learning, and there will be a facility to reply/comment underneath the blogposts, so it can support some discussion. Please sign up at **https://blog.foster77.co.uk/** now and take a look at the first post on 14 April.