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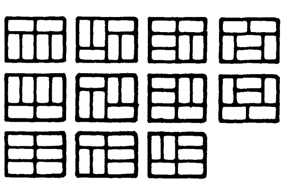
I found the following investigation, from the new SMP 16–19 course, very interesting:

> "**In how** many different ways can you fill a rectangle measuring m units by n units with tiles **shaped** like dominoes which are 2 units long and 1 unit wide?"

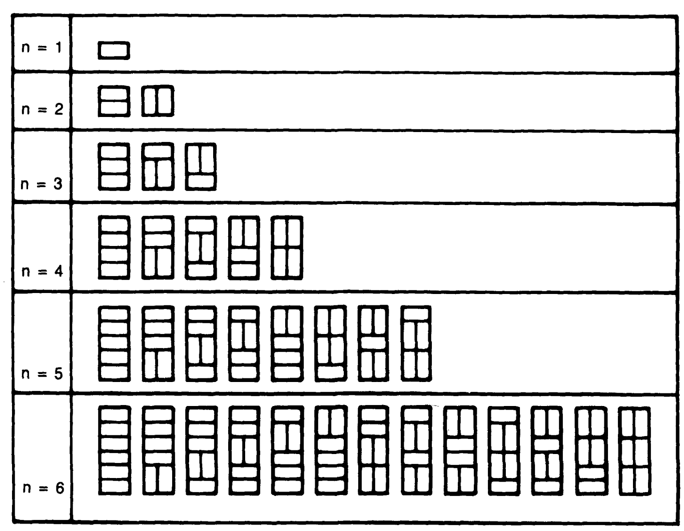
To illustrate what this means, there are exactly 11 ways of tiling a 3 by 4 rectangle in this way, as the diagram shows.



The investigation in this generality is a very hard problem, beyond sixth form level, but it does contain some interesting maths, and simple cases ("How many ways of tiling a 2 by 10 rectangle?") would definitely make a good investigation for younger pupils, with scope for work at all levels. I gave the problem of tiling rectangles 2 units wide to a top set year 9 class who had no problem spotting the number pattern. The task of drawing all possible tilings of a given rectangle, as here with a 3 by 4 rectangle, making sure none have been missed, is accessible to most pupils. It is impossible to tile for instance a 3 by 3 rectangle like this, or any other shape with *odd* area. Each domino has area two, so the total area covered by a tiling must be *even*.

A related puzzle is to tile a chessboard which has had two diagonally opposite corner squares cut off, using the same 2 by 1 dominoes. The chessboard certainly has an even number of squares, but still the tiling is impossible. The reason for this is just a different sort of parity. The corners cut away are the same colour, black in this case, so the truncated chessboard has *more* white squares than black. But, as each domino must cover exactly one white square and one black square, any tiling must cover *the same* number of white squares as black, and therefore it is impossible.

Taking the simplest case of rectangles 2 units wide and n units high, it is quite easy to work out the first few terms of the number pattern.



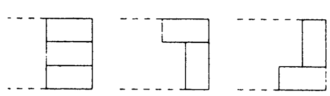
For $n=1, 2, 3, 4, 5, 6$ the number of different tilings are 1, 2, 3, 5, 8, 13. This is the familiar *Fibonacci sequence.* The problem now is to explain why? To answer this, I used some recurrence relation notation. Let u_n be the number of ways of tiling the 2 by n rectangle. So, $u_1=1$, $u_2=2$, $u_3=3$, $u_4=5$, $u_5=8$, ...

Each tiling of a 2 by n rectangle must end either with a vertical domino, or with two horizontal dominoes. So, each of the tilings in the first case comes from a 2 by $(n-1)$ rectangle with a single vertical domino added on the edge; there are u_{n-1} of these altogether. Each of the tilings in the second case comes from a 2 by $(n-2)$ rectangle with a pair of horizontal dominoes added on the edge; there are u_{n-2} of these. Putting the two possibilities together gives the familiar recurrence relation for the Fibonacci sequence:

$$u_n = u_{n-2} + u_{n-1} \text{ with } u_1 = 1 \text{ and } u_2 = 2.$$

Intuitively, solutions are built up from shorter solutions by adding either one vertical or two horizontal dominoes to the end. So you can build from any of the previous two smaller solutions to get the next.

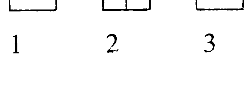
So far so good. Time to move on to 3 by n rectangles. We already know that only rectangles of even length will be possible this time, and that there are 3 ways of tiling the 3 by 2 rectangle, 11 ways of tiling the 3 by 4 rectangle. In fact, it is possible to set up a recurrence relation in almost the same way as for 2 by n rectangles. Whereas before, there were two possible endings, a single vertical domino or two horizontal tiles, now there are three endings to consider, corresponding to the three different ways of tiling a 2 by 3 rectangle:



a_n b_n c_n

Let the number of tilings of a 3 by 2n rectangle with each of these endings be a_n, b_n and c_n respectively. So, the total number of tilings of a 3 by 2n rectangle, u_n, is given by

$$u_n = a_n + b_n + c_n$$

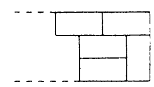


1 2 3

By symmetry, of course, $b_n = c_n$, so we will be able to simplify things later on. Now, just as before, I will look at how we can build up from a smaller tiling. This time, we will build up by adding one of three different blocks to each of the three possible endings. Each block is to be two units wide as there are no tilings of odd length. Block 1 can be added to any of the endings in precisely one way, and will generate a longer tiling with ending a), and this will account for all possible tilings of this length with this ending. So,

$$a_{n+1} = u_n = a_n + b_n + c_n = a_n + 2b_n$$

Block 2 can be added in the same way to each of the endings to generate a longer tiling with ending b), but there is another possibility. If block 2 is added to ending b), the adjacent vertical dominoes that result can be flipped to produce a new sort of tiling:



This has to be incorporated into the recurrence relation for b_n, so we get an extra term:

$$b_{n+1} = u_n + b_n = a_n + 3b_n$$

Now we're there! The final recurrence relation is:

$$u_n = a_n + 2b_n \text{ where } a_{n+1} = a_n + 2b_n,$$

$$b_{n+1} = a_n + 3b_n \text{ and } a_1 = b_1 = 1$$

To check, $a_2=3$, $b_2=4$ so $u_2=11$ gives the correct number of tilings of the 3 by 4 rectangle.

For 4 by n rectangles, it gets harder – but still the recurrence method can be made to work. Of course, all of this is more complex than the investigation asked for, but I have been impressed by the amazing range of levels that this task can be approached – from year 9 to degree level.

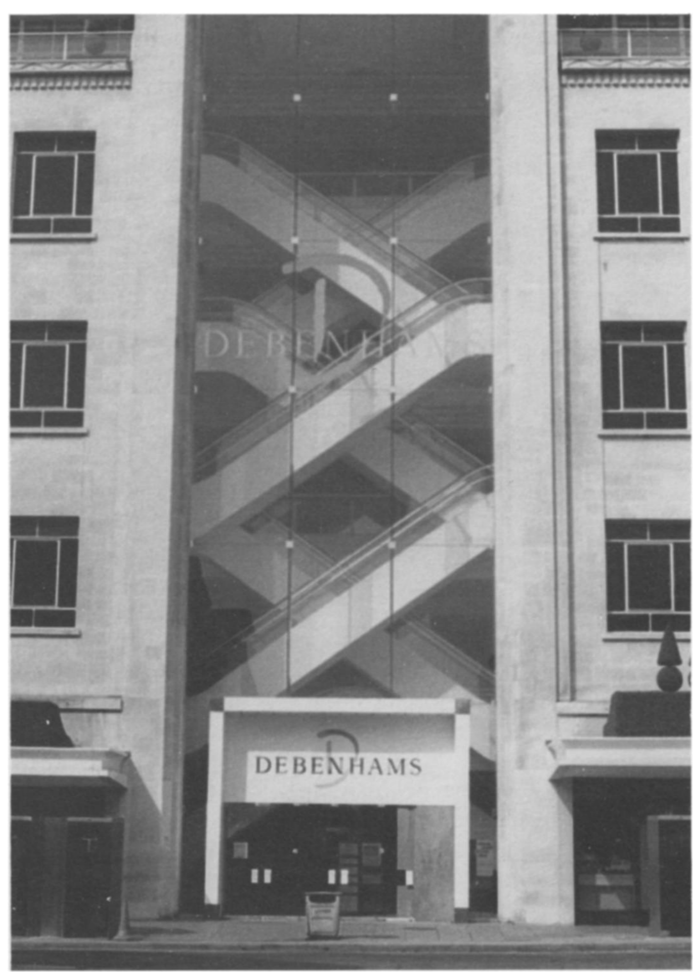


Photo: Malcolm Knee