COUNTING IN CUNEIFORM

by
Eleanor
Robson

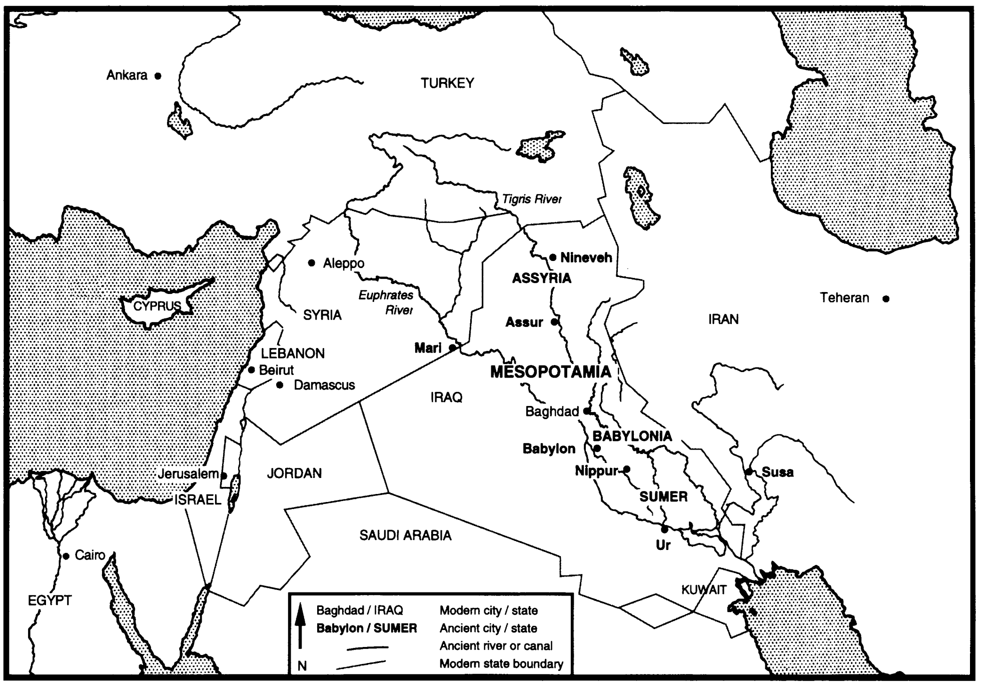


Fig. 1
Map of the ancient Near East

Introduction

Resources for the study of ancient Egypt are easily available, for all levels of education. But the teacher who is interested in its northern neighbours in Mesopotamia is hard pushed to find anything other than undergraduate textbooks. And if that teacher is particularly interested in Mesopotamian (more often called Babylonian) maths, there is less still. The activities described here are an attempt to redress that balance, based on original research and ancient artefacts, and on several workshops run in recent years. They can be adapted for use with many different groups, from primary school children upwards, for a variety of mathematical and numerical topics.

Mesopotamia, 'the land between two rivers', more or less covers the area of modern Iraq (see Fig. 1). It was home to a series of complex and influential civilizations—Sumerian, Babylonian, Assyrian—over a span of three thousand years, until its gradual demise through repeated conquests by the Persians, Greeks, and Parthians in the second half of the first millennium BC (see Fig. 2). What the great Mesopotamian cultures all had in common was a complicated syllabic wedge-based, or 'cuneiform', script, with which trained scribes wrote on clay tablets. These were for the most part the humdrum administrative documents which large institutions continue to generate today: memos, receipts, wage records, and the like. But more exciting texts survive too, including magical rituals, law codes, vivid descriptions of military campaigns and battles, and great myths such as the *Epic of Gilgamesh*. Because clay, unlike papyrus or paper, does not decay, hundreds of thousands of tablets survive, from many different periods and places in Mesopotamia, allowing for an unprecedentedly detailed view of some aspects of its history—assuming, that is, the historian does not get overwhelmed by the sheer quantity of them all.

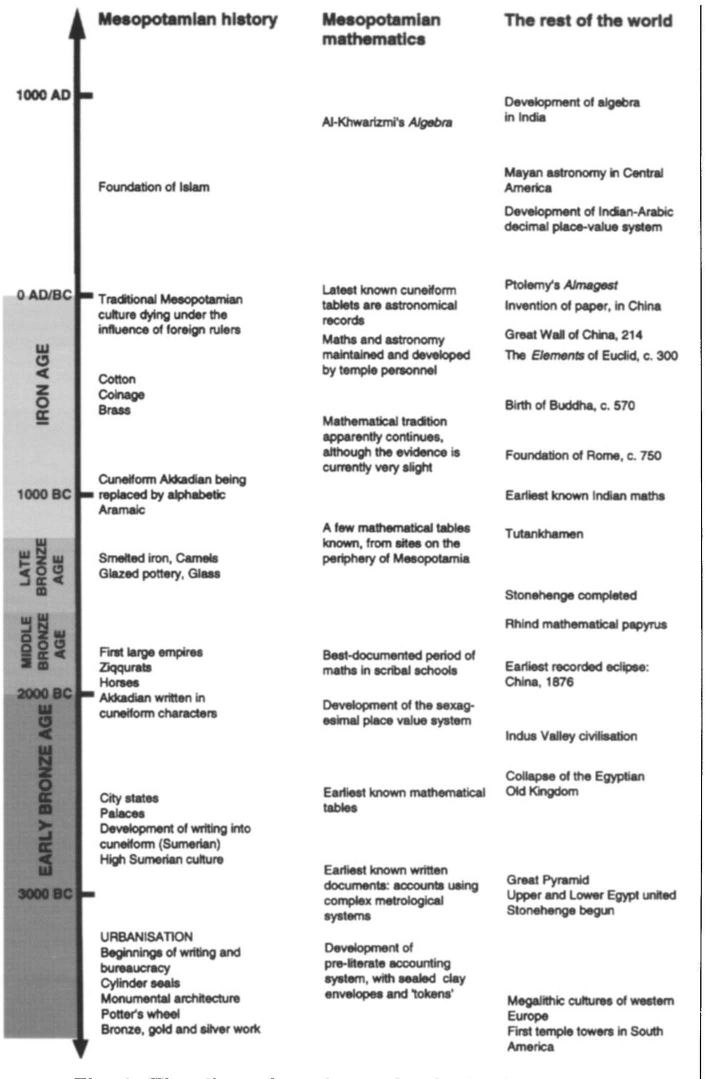


Fig. 2 Timeline of mathematics in the Near East

All of these works, from dockets to literary epics, were written in the now long-dead languages of Sumerian (which has no known linguistic relatives) and Akkadian (from the same language-family as Hebrew and Arabic), or a combination of the two. Because the cuneiform script used to write these languages was highly complex, literacy was confined to a professional body of trained scribes. Their education naturally contained a large component of numeracy and mathematics, and many of their educational materials, particularly from the early second millennium BC, or 'Old Babylonian period', have survived. They form the basis of our understanding of what most modern textbooks call 'Babylonian' mathematics. The Old Babylonian base 60, or 'sexagesimal', place-value system survived long after the demise of Mesopotamian civilization in the records and calculations of the Classical, Islamic, Indian, and early modern European astronomers. It is the distant ancestor of the modern division of the hour and the degree of arc.

There are two parts to the material presented here: background information for the teacher, followed by suggestions for classroom activities. These try to recreate the experiences of the trainee scribes, as they learned to record cuneiform numerals in clay, and to perform the basic arithmetical operations. There are many aspects of scribal school life we still know nothing about—how long schooling lasted, how old the pupils were, criteria for entering and leaving scribal school—and others we can only guess at—the size of the classes, the order of the curriculum, the sort of people who went to school. Much of what follows, then, is based on hypothesis and disparate pieces of research published in obscure specialist journals. While there is a list of recommended reading and resources at the end, be warned that there is (at least not yet) no easy way in to this subject.

Writing Cuneiform Numerals

For each child you will need a generous handful of plasticine or modelling clay and a stylus. The Mesopotamian scribes used lengths of reed. I make my own with quadrant dowelling bought from a DIY superstore, cut it up into 6-inch lengths and sand the ends to remove splinters. Cheap wooden chopsticks are just as good, or lengths of square-sectioned garden canes. But it doesn't really matter what you use, as long as it is comfortable to hold like a pencil, and it has a right-angled corner at the business end.

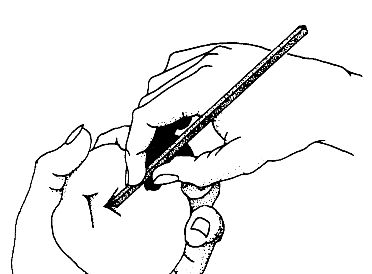


Fig. 3 Writing a cuneiform tablet

Making cuneiform wedges uses a very different technique to writing on paper, because the end result is a three- not two-dimensional object. For a start, you need to hold the lump of clay in your hand and not flatten it out onto a table. Because you can move both tablet and stylus it makes no difference whether you are left- or right-handed. Hold the stylus no more than about 30° to the surface of the clay, with a long edge pointing downwards. Press the end of that edge (about 5–10 mm), corner first, into the tablet and lift up again, without rotating or dragging the stylus. You will be left with a long narrow triangular impression, with the deepest point near the narrow side (the 'head') and the shallowest at the apex of the triangle (the 'tail') (Fig. 3). A single wedge with the tail pointing to the reader represents the numeral 1.

The 10 numeral is written at a 30–45° anticlockwise angle to the 1—you can turn the tablet or stylus or both—and with the hand rotated so that the flat surface of the stylus to the right of the long edge makes much more contact with the clay. Approximately equal lengths of the top edge and the long edge are impressed, so that the wedge looks like a symmetrical arrow-head pointing left (Fig. 3). As your technique improves—and I strongly suggest you practise before introducing cuneiform to your class—you will be able to make smaller and more elegant wedges, however large the stylus is. The unit wedge is repeated in up to three rows of three marks to form the numerals 2–9; the ten is aligned diagonally in threes to write 20, 30, 40, and 50 (see Fig. 4).

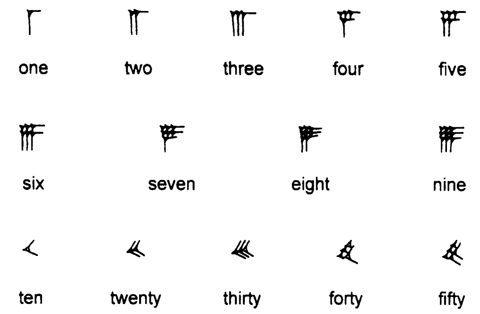


Fig. 4 The sexagesimal numerals 1–9 and 10–50

The Sexagesimal Place Value System

These numerals can be combined into a place value system, with the larger places ranged to the left of the smaller. Thus the units can in fact represent any power of sixty: 1×60^n, where n is any integer; and the tens may stand for any 10×60^n. Thus 4 tens to the left of 5 units could be read as 45, or $45 \times 60 = 2700$, or $45/60 = 3/4$, or 45 times any power of sixty, as big or as small as we like. Conversely, 4 tens to the *right* of 5 units could be read as $5 + 40/60 = 5\ 2/3$, or $5 \times 60 + 40 = 340$, or $5/60 + 40/3600 = 1/12 + 1/90$ (0.09444), or $5 \times 3600 + 40 \times 60 = 20{,}400$, etc. In modern transliteration the 'sexagesimal point' is marked with a semicolon and a space separates each power of sixty (see Fig. 5).

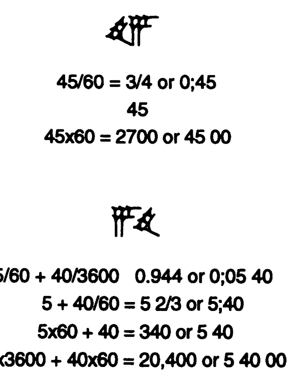


Fig. 5 45×60^n and $5;40 \times 60^n$

Although we show zeros in transliteration there is no cuneiform sign for zero, in any of its roles. There was no need to mark single empty tens or unit places. To show a completely empty sexagesimal place with no tens or units—to distinguish 602 = 10 02 from 12, say—the scribes just left a space on the tablet. In later times a space marker of two small tens wedges above each other was adopted (see Fig. 6). But

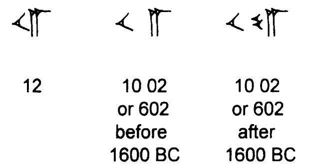


Fig. 6 Ways of marking medial zero

there was no way at all to show absolute value: no equivalent of the zeros after the decimal point or at the right hand side of number which we use to indicate the size of a number. This could be seen as a major flaw in an otherwise elegant, minimalist and efficient calculation scheme, but is explicable in the light of its restricted function: the sexagesimal place-value system was used solely for professional calculation.

Ancient School Arithmetic

Various absolute-value systems had been used to record everyday counting and measuring since proto-historic times (see Fig. 7). Rather like the pre-metric metrological systems of Europe and north America, they used a mixture of units—30 fingers in a cubit, 12 cubits in a rod, for instance (see Fig. 8)—which were inevitably complicated to manipulate arithmetically. For this reason, the scientific sexagesimal system was invented, some time around the end of the third millennium BC, to shortcut the complicated procedures needed for multiplication and division in particular. It is difficult to pinpoint exactly when this happened because it was intended—and used—merely as an aid to computing, and results were converted back into everyday measure again. Thus we find the sexagesimal place-value system used only in school tablets whose point was to train scribes in its use, or on professional documents from which the (trained) scribe had omitted to erase his rough work.

We can see the workaday–scientific–workaday conversion principle at work in several school pupils' exercises from the ancient city of Nippur. In the example shown (Fig. 9), the problem is set in the bottom right hand corner, and the remains of the rough work are shown at the top left. The text, in Sumerian with a loose and a literal translation, reads:

2 shu-si íb-si$_8$	The side of a square is 2 fingers.	2 fingers square-side
a.shà-bi en-nam	What is its area?	area.its what.is
a-shà-bi igi-3-gál she-kam	Its area is a third of a grain.	area.its third grain.is

The syllables of the writing system do not always correspond to grammatical particles. Sumerian is still badly understood, as it is related to no other known language, lexically, phonetically, or grammatically and has to be interpreted for the most part through the filter of other long-dead languages such as Akkadian. Technically, it is classified as an agglutinating ergative isolate.

The first stage in our calculation is to convert the length measured in fingers into a sexagesimal fraction of the standard length unit, the rod (see Fig. 8).

2 fingers = 2/30 = 0;04 cubits = 0;04 ÷ 12 rods = 0;00 20 rods.

Traces of this resulting 20 can just be seen at the edge of the break on the top left of the tablet. A second copy of the 20 is written immediately underneath, so that it can be multiplied with itself to get the area of the square, measured in 'gardens':

$$0;00\ 20 \times 0;00\ 20 \times 0;00\ 00\ 06\ 40 \text{ 'gardens'}.$$

We can then multiply this product by 60 (shekels in a 'garden') and 180 (grains in a shekel) to get 0;20 grains of area—the answer given on the tablet.

So, we can see that the sexagesimal numbers are used only

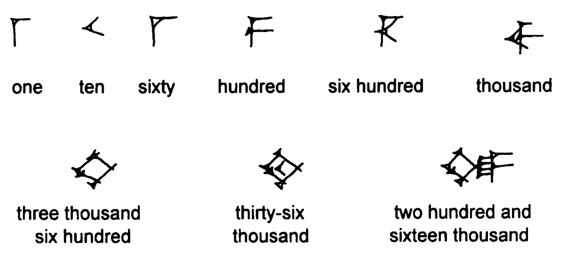


Fig. 7 Absolute number notation in cuneiform

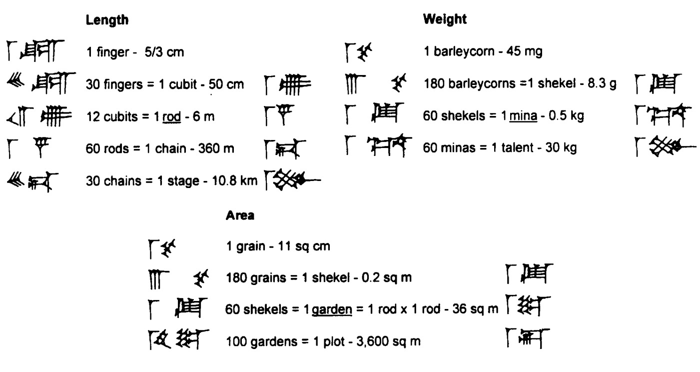


Fig. 8 Some Mesopotamian weights and measures

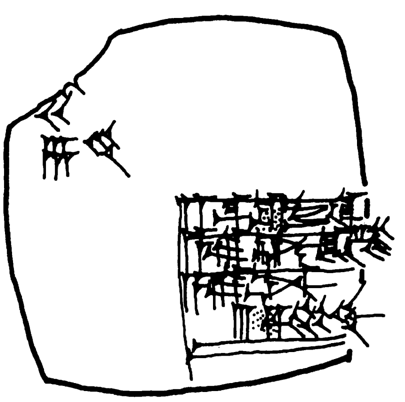
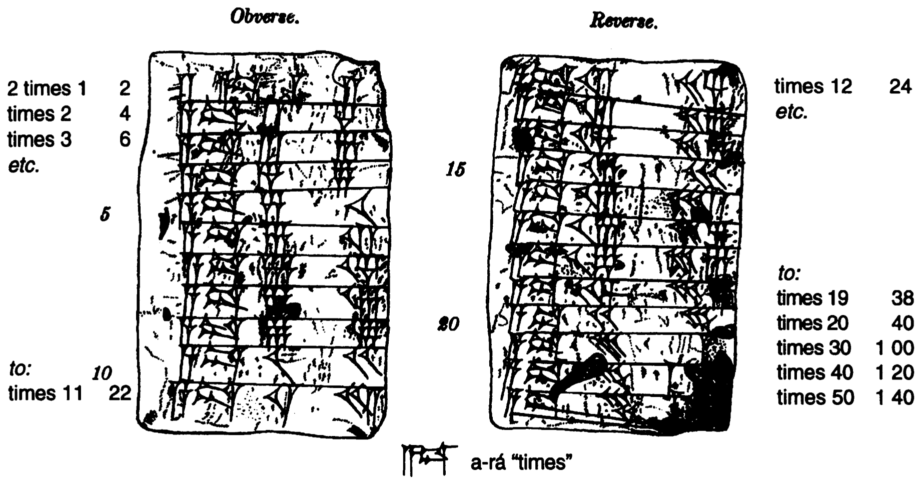


Fig. 9 A metrological calculation



There are multiplication tables for:

50	48	45	44 26 40	40	36	30	25	24	22 30		
20	18	16 40	16	15	12 30	12	10	9	8 20		
8	7 30	7 12	7	6 40	6	5	4 30	4	3 45		
3 20	3	2 30	2 24	2 15	2	1 40	1 30	1 20	1 15		

Fig. 10 A multiplication table, front and back

in the workings and not shown in the final answer. In other words, their absolute value is not important as long as the scribe is sensible about the size of answer he expects—rather as we ignore decimal points during the intermediate steps of long multiplication and long division.

To help them in their arithmetic the trainee scribes learned a large standard set of tables, first for converting between base 60 and the various metrological systems and then for multiplication (Fig. 10) and division—which involved finding the sexagesimal reciprocal of the divisor and then multiplying (Fig. 11). They learned first by copying a teacher's model and then by repeated writing until they had the tables off by heart. Many hundreds of these practice copies have survived, especially from Nippur. The tablet may contain just one single table or anything up to the whole standard set. There is often a non-mathematical school exercise on the other side of the tablet. We also have examples of arithmetical rough work created as the students solved word problems. Typically they involve repeated multiplications and divisions of one- or two-place sexagesimal numbers (see Fig. 12). The evidence

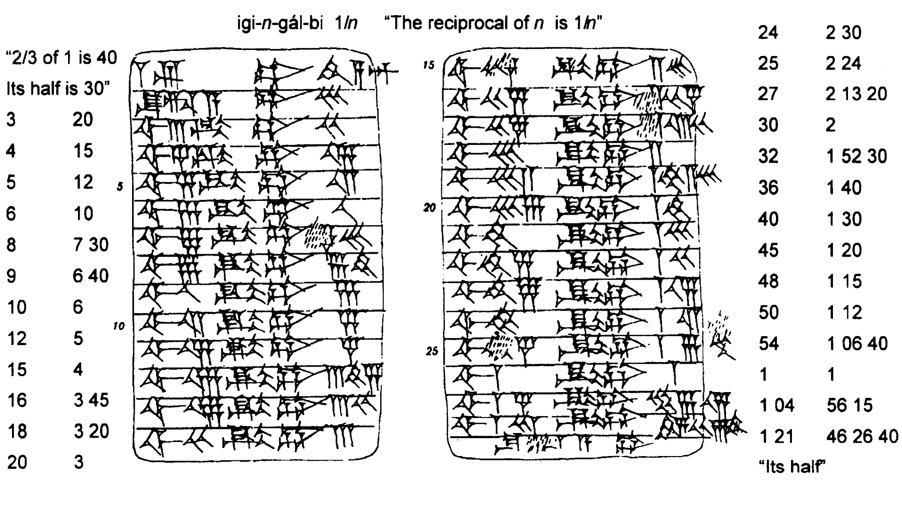


Fig. 11 A standard reciprocal table

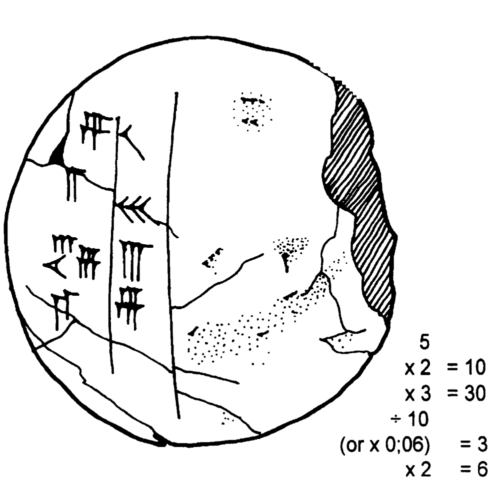


Fig. 12 A school exercise in repeated multiplication and division

now is that arithmetic was taught in the third phase of the elementary curriculum at Nippur. Sadly, though, we still know nothing of how old the students were at the time, or how long this or any other part of their schooling took.

Modern Classroom Activities

These suggestions for classroom activities are in the form of photocopiable work cards, with selected outline answers given below. Clearly not all the topics suggested here will be appropriate for your pupils, and you will almost certainly want to create your own alternatives to my suggestions and to add greatly to the numerical examples.

Selected outline answers to the work cards

1. The pattern is three rows of three. Modern numerals are further abstracted from their associated number; you couldn't tell from looking at them which numeral they represent if you didn't already know. Clay was abundant and cheap, paper not yet invented, and papyrus grew only in Egypt. Methods of recording number include keeping tallies, using accounting tokens, pebbles or shells, knotted strings (quipus), electrical pulses (computers), etc.
2. We still count minutes and seconds of time and angle in sixties. We also count in pairs, dozens, reams, etc. We can't write six tens because we have to move up a sexagesimal place. Use the 1 sign again, to the left of the tens.
3. There is no 'largest number' in cuneiform. We leave a space between the 1 and 2 signs to distinguish 62 from 3.
4. The Mesopotamians thought that 'nothing' on the tablet was a sensible representation of zero. They had to simply remember 'medial zeros', until the sign shown in Work card 5 was invented.
5. Some fractions (with divisors other than 2, 3, 5) are non-terminating. We can therefore only write approximations to them.
6. Work out $3 \times 1\ 00$, $\times\ 30$ and $\times\ 4$, and add. Or add 3×100, $\times\ 40$ and subtract $\times\ 6$, etc.
7. Scribes needed multiplication to work out areas and volumes of all sorts, to calculate wages and interest, to estimate crop yields, etc.
8. The left-hand numbers increase as the right-hand ones decrease; all numbers are sexagesimally regular (i.e. they have factors of 2, 3, and 5 only); left-hand numbers are integers 1–60 plus squares of 8 and 9 (other squares 1–10 are already included); none in right-hand column has more than three sexagesimal places, etc. Irregular numbers such as 7 are omitted. There are a total of 64 one-, two-, and three-place pairs of sexagesimal reciprocals.
9. One way of dividing by a number that wasn't on the reciprocal table would be to halve and double (or similar with 3s or 5s) a related reciprocal pair until you got the number you wanted. The scribes would have needed division to allocate rations and wages, portions of fields, canal-water for irrigation, to calculate bartering rates, and to calculate the labour and materials needed to build or repair canals and walls.
10. $5;15 \times 5;15 \times 27;33\ 45$. A square—correct.
11. $54 \times 57;30 \div 2 = 25\ 52;30$. Finding the area of an isosceles triangle—correct.
12. Cuneiform eventually died out under pressure from alphabets like Greek and Aramaic. Persia, Greece and Egypt, and later Rome and the Sasanian empire, borrowed all sorts of Mesopotamian ideas. The base 60 place-value system flourished in astronomy for a millennium after cuneiform. Some geometrical methods may have survived into the work of Heron and Islamic mathematics, but this transmission is less well documented. The Classical Mediterranean, including Turkey and Egypt, and then Islam and India, were full of mathematical activity in the first 1500 years AD.

Further Reading and Resources

Background reading for teachers on mathematics in Mesopotamia

Aaboe, A. 1964 *Episodes from the Early History of Mathematics* (New mathematical library; 13), Mathematical Association of America. ISBN 0883856131.

Høyrup, J. 1994 'Babylonian Mathematics'. In Grattan-Guinness, I. (Ed.), *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, Vol. I, pages 21–29. ISBN 0415037859.

Høyrup, J. 1992 'Mathematics, Algebra, and Geometry'. In Freedman, D. (Ed.) *The Anchor Bible Dictionary*, Vol. IV, pages 601–612, Doubleday. ISBN 0385193629.

Neugebauer, O. 1969 *The Exact Sciences in Antiquity* (2nd Edn.), Dover. ISBN 0486223322.

Nissen, H. J., Damerow, P. and Englund, R. K. *Archaic Bookkeeping: Early Writing and Techniques of Economic Administration in the Ancient Near East*, University of Chicago Press. ISBN 0226586596.

Postgate, J. N. 1994 *Early Mesopotamia: Society and Economy at the Dawn of History*, Routledge. ISBN 0415110327.

Roaf, M. 1990 *Cultural atlas of Mesopotamia and the Ancient Near East*, Facts on File. ISBN 0816022186.

Walker, C. B. F. 1987 *Cuneiform* (Reading the past), British Museum Press. ISBN 0714180599.

Classroom books on Mesopotamia

Eagle, M. R. 1995 *Exploring Mathematics Through History*, Ch. 1, Cambridge University Press. ISBN 0521456266.

Hunter, E. C. D. 1994 *First Civilizations* (Cultural atlas for young people), Facts on File. ISBN 0816029768.

Lumpkin, B. 1995 *Multicultural Math and Science Connections*, J. Weston Walch.

Moss, C. 1989 *Science in Ancient Mesopotamia*, Franklin Watts.

Nuffield Advanced Mathematics 1994 *History of Mathematics*, Longman. ISBN 0582257298.

Web sites

Mesopotamian Mathematics, by Duncan Melville at St. Lawrence University: http://it.stlawu.edu/dmelvill/mesomath/index.html

Recommended Reading on the Ancient Near East: Ancient Mesopotamia, by Carole Krucoff, Oriental Institute Museum Education Department, University of Chicago: http://www-oi.uchicago.edu/OI/DEPT/RA/RECREAD/mesobib.html

Radio programmes

Programme 1 of *The Square on the Pythagoras*, a 4-part series on the history of mathematics. Producer A. McNaught. First broadcast on BBC Radio 4, October 1995.

Programme 1, on early Mesopotamia, of a 20-part series *Civilization*. Producer M. Diamond. First broadcast on the BBC World Service, February 1998.

UK Museums displaying Mesopotamian mathematical clay tablets

The British Museum, Great Russell Street, London WC1B 3DG, in the Sackler Gallery of Early Mesopotamia. Tel. 0171 323 8599 http://www.british-museum.ac.uk/

The Ashmolean Museum, Beaumont Street, Oxford OX1 2PH, in the Drapers' Gallery of the Ancient Near East. Tel. 01865 278000 http://www.ashmol.ox.ac.uk/

Acknowledgements

This article arose from a series of workshops, given initially at the Maynard School, Exeter, at the invitation of John Earle in June 1996, and repeated at the British Association of Near Eastern Archaeology annual conference in Oxford, December 1996, the British Society for the History of Mathematics History of Maths in Education meeting in Sheffield, February 1997 (with John Earle), the Association of Portuguese Mathematics Teachers annual conference in Figuera da Foz, Portugal, November 1997, and the National Mathematics Days in Noordwijkerhout, the Netherlands, in January 1998. It is a great pleasure to thank John Earle, both for initializing the project and for overseeing its eventual publication. It is equally pleasurable to thank the many conference organisers and enthusiastic workshop participants for their warm hospitality and encouraging and constructive feedback. All shortcomings and errors in this paper are, of course, my own.

UET 6/2 233, from Old Babylonian Ur (Fig. 12), is published by permission of the Trustees of the British Museum. CBS 7265 (Work Card 10), UM 29-15-192 (Fig. 9), and UM 29-15-195 (Work Card 11), all from Old Babylonian Nippur, are published by permission of the Babylonian Section of the University Museum, Philadelphia. All drawings are the author's originals, except Fig. 10 (HS 222a, from Old Babylonian Nippur, after Hilprecht, *BE* 20/1 (1906), pl. 1) and Fig. 11 (MLC 1670, unprovenanced, after Clay, *BRM* 4 (1923) pl. 37).

Author
Eleanor Robson, Wolfson College, Oxford

Counting in Cuneiform Work Card 1

Writing numbers in cuneiform

You can write a 'one' in cuneiform by pressing your stylus into the tablet and lifting it up again. Don't drag it along the clay or twist it. If you make a mistake you can rub it out with your thumb! Keep practising until you are confident.

- How might we write two?
- What number is this? 
- And this? 
- What's the pattern? Write all the numerals 1–9 in cuneiform.
- Do modern numerals make a pattern like this? Which is more sensible, in your opinion?
- Why do you think Mesopotamian scribes wrote numbers in clay and not on paper? How many other methods of recording numbers can you think of?

Counting in Cuneiform Work Card 2

Bigger numbers

You now know how write the numerals 1–9 on a clay tablet. We could continue this pattern but it would become very clumsy, so we introduce a new sign for 'ten'. Turn your tablet or stylus 30°–45° and make a wedge that looks like an arrow-head facing left. Practice for a bit.

- Now guess what 20 looks like.
- What's this number? 
- And this one? 
- Choose two other numbers between 1 and 50 for your neighbour to write. Did they get them right?

The Mesopotamians counted in 60's.

- Do we still count or measure anything in 60's?
- What about other numbers?
- How could you write 60 on your tablet? You're not allowed to write six tens! (Why not?)

Counting in Cuneiform Work Card 3

Place-value systems

The Mesopotamians used a place-value system to write numbers: numerals to the left are a power of sixty bigger than the ones on the right.

- Does our modern number system use place value?
- What is this number? 
- What does 753 look like in base 60 cuneiform?

We write base 60 in modern notation like this, with a space separating each sexagesimal place:

 44 26 40 (160,000)

- Choose two other large numbers for your neighbour to write in base 60 cuneiform. Check that they get them right.
- What is the largest number you can write in cuneiform?
- How can you distinguish between 62 (or 1 02) and 3 in cuneiform?

Counting in Cuneiform Work Card 4

Zero and negative numbers

In cuneiform, you leave a space on the tablet to show a blank sexagesimal place.

- Do you think this is a sensible idea? Why?
- What happens if the zero needs to come at the end of a number, like 12 00?
- Invent a cuneiform sign for zero, and explain why you chose it.
- Write 432,000 in base 60 cuneiform.

There is no way of marking negative numbers in cuneiform. If you want to deal with numbers less than zero you will need to invent a 'negative' sign as well.

- What sign would you choose for negatives? Why?
- Set your neighbour some simple additions and subtractions to do in cuneiform. Check their answers.

Counting in Cuneiform Work Card 5

Very small numbers

You've seen that you can write numbers as big as you like in cuneiform. How could you write very small numbers without inventing new numerals?

- What is half of 60?
- What is a third of 60?
- What fraction of 60 is 8?

If we use to mean no whole numbers,

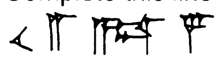
- What is this number?
- Give its decimal and fractional equivalents.
- What is this number?
- Give its decimal and fractional equivalents.
- Ask your neighbour to write some other fractions.
- Which fractions can't you write in cuneiform? Why?

Counting in Cuneiform Work Card 6

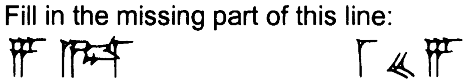
Multiplication tables

The Mesopotamian scribes learned their tables off by heart. The sign for 'times' is but there is no sign for 'equals'.

- Write a three-times table up to 20 in cuneiform.
- Complete this line from a multiplication table:



- Fill in the missing part of this line:



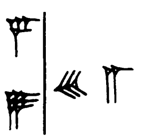
Cuneiform times tables ended with multiplications by the numbers 20, 30, 40, and 50.

- Add the lines for 30, 40, and 50 to your three-times table. What pattern do the answers make?
- Work out 3 x 1 34 using your table. How did you do it? Was your method the same as your neighbour's?

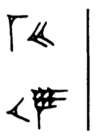
Counting in Cuneiform Work Card 7

Doing multiplication

The Mesopotamian scribal students did not write 'times' every time they did a multiplication. Instead, they drew a vertical line on their tablet, and put the numbers to be multiplied on the left of it and the answer to the right, like this (4 x 8 = 32):



- Try this multiplication:



- Write out and solve 23 x 14 in base 60 cuneiform.
- Give your neighbour a difficult multiplication to do. Check their answer.
- Why did Mesopotamian scribes need to multiply? What practical uses does it have?

Counting in Cuneiform Work Card 8

Reciprocals

'Reciprocal' is another word for the inverse of a number.

- What is the answer when you multiply a pair of reciprocals together?
- What is the modern base 10 reciprocal of 3?
- What is the base 60 reciprocal of 3?
- Does every number have a reciprocal?

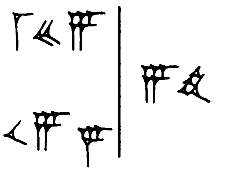
Look at the pairs of reciprocals in Fig. 11.

- What number patterns can you detect on the tablet?
- Why is 7 missing?
- Which other numbers are missing? Why? Should any be added to the table?
- Why do you think that 1 04 and 1 21 have been included at the end of the table?

Counting in Cuneiform Work Card 9

Dividing

The Mesopotamian scribes thought of division as multiplication by a reciprocal. To divide 1 25 by 15, for instance, they would find the reciprocal of 15 and then multiply it by 1 25. They wrote the reciprocal next to its pair:



1 25 ÷ 15 =
1 25 x 0;04 =

1 00 x 0;04 = 4
+ 25 x 0;04 = 1;40
= 5;40

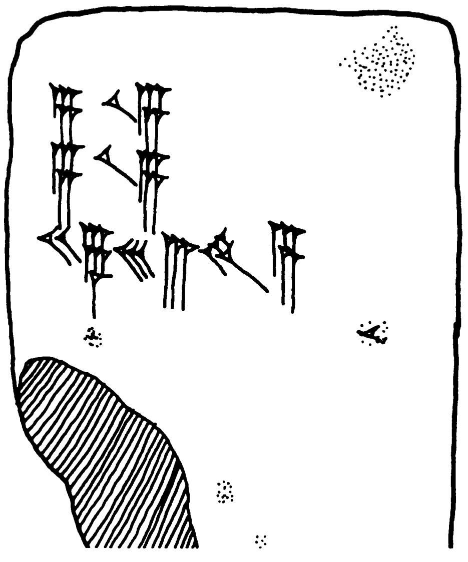
- Divide 1 04 by 16, by finding the reciprocal.
- Choose a number on the standard reciprocal table, and use it to set a division problem for your neighbour. Check their answer.
- How could you divide by a number that wasn't on the reciprocal table?
- When would the Mesopotamian scribes have needed to do division? What practical uses does it have?

Counting in Cuneiform Work Card 10

Deciphering a tablet (1)

This tablet was written nearly 4,000 years ago by a trainee scribe.

- Can you read the numbers?
- What problem was the scribe trying to solve? Did they get it right?

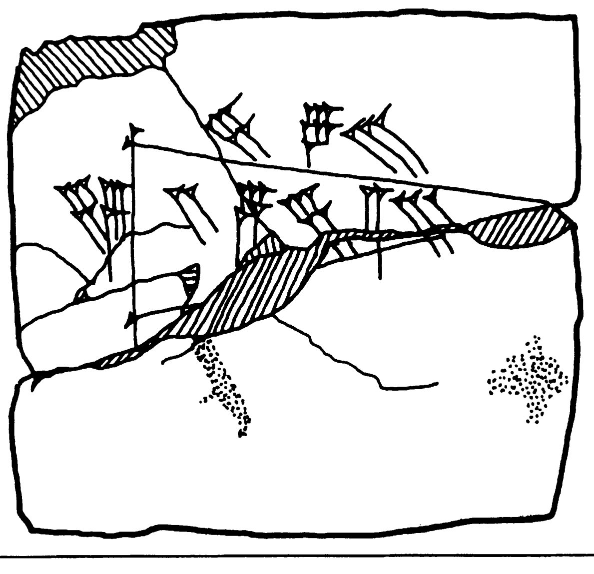


Counting in Cuneiform Work Card 11

Deciphering a tablet (2)

This tablet was written nearly 4,000 years ago by a trainee scribe.

- Can you read the numbers?
- What problem was the scribe trying to solve? Did they get it right?



Counting in Cuneiform Work Card 12

After cuneiform

People stopped writing cuneiform about 2,000 years ago.

- Why might this have happened?
- Which neighbouring cultures might have borrowed mathematics and other ideas from Mesopotamia?
- Which parts of Mesopotamian mathematics survived after cuneiform?
- Was the Middle East an important place for mathematics after cuneiform?
- Find out what you can about the rediscovery of ancient Mesopotamia and cuneiform.