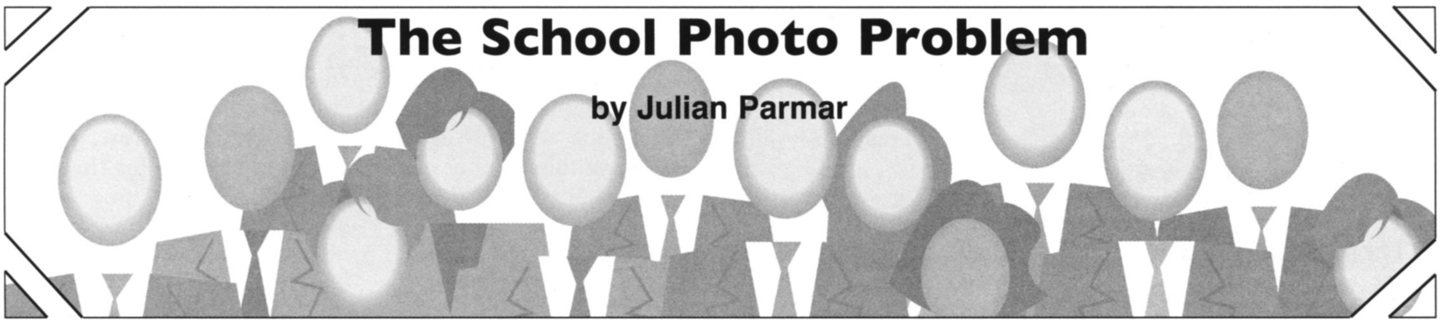
The School Photo Problem

by Julian Parmar



A few months ago, a group of friends and I attended a Maths Conference run by teachers from our school to broaden our mathematical experience, and hopefully to have some fun as well! We found the day very rewarding, learning new techniques to solve a multitude of different types of problems by working through some examples, and then working in groups of four or five to solve a sheet of questions. Another valuable aspect to the conference was that we were given a talk on studying Engineering at university, which many of us found very interesting. Towards the end of the day, one of our teachers threw us an open problem, challenging us to find a solution. In the half-hour that we had, we came up with all sorts of random formulae (mostly through guesswork!), and didn't get very far. On returning home, I still found the unsolved problem annoying me, so I sat down for a few hours and created a solution. I found it to be a very interesting problem, one of those which seems simple, but for which it is difficult to find a solution. I'd like to present it here, not as a rigorous proof, but rather as a piece of interesting mathematics. I hope that you enjoy reading my solution as much as I enjoyed exploring it!

The Problem

The arrangement for a school photo containing an even number of pupils is as follows:

1. The pupils must be in two rows.

2. The pupils must be arranged in increasing height order from left to right.

3. A pupil on the back row must be taller than his/her corresponding pupil on the front row.

How many arrangements are there for *2n* (an even number of) pupils?

The Solution

The first hurdle was to find a way of expressing the formula in a mathematical way that I could manipulate easily. I decided to use a *matrix* form, where each a_i represents a pupil, as follows:

$$\begin{bmatrix} a_2 & a_4 & \cdot\cdot & a_{2n} \\ a_1 & a_3 & \cdot\cdot & a_{2n-1} \end{bmatrix} \text{where } a_i \in Z^+.$$

In order to consider this problem, only the bottom row needs to be used, as any bottom row arrangement gives a unique top row arrangement, containing all the a_i which are not in the bottom row arranged in ascending order from left to right. The starting number on the bottom row is always 1, as it is the smallest number.

To be able to spot a relationship, let us only consider the first k columns, denoted by c_k.

Note that however many columns there are in the complete matrix, the first k columns can only be arranged in a certain number of ways. Here are the possibilities for c_2 and c_3.

c_2

$$\left.\begin{matrix} [1\ 2\ .\ .] \\ [1\ 3\ .\ .] \end{matrix}\right\} 2$$

Note:

$[1\ 4\ .\ .]$ would not work, as 2 and 3 would have to go on the top.

$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ which puts 3 behind 4.

This means that for a matrix of any size, the first two columns can only be arranged in two ways.

c_3

$$\left.\begin{matrix} [1\ 2\ 3\ .\ .] \\ [1\ 2\ 4\ .\ .] \\ [1\ 2\ 5\ .\ .] \end{matrix}\right\} 3$$

$$\left.\begin{matrix} [1\ 3\ 4\ .\ .] \\ [1\ 3\ 5\ .\ .] \end{matrix}\right\} 2$$

Note:

$$[1\ 2\ 6\ .\ .] \rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

and

$$[1\ 3\ 6\ .\ .] \rightarrow \begin{bmatrix} 2 & 4 & 5 \\ 1 & 3 & 6 \end{bmatrix}$$

would not work, as positioning the other numbers on the top row would place 5 behind 6.

Each matrix can generate a set of matrices which all start with the same pattern, but end differently. If we look at the number of permutations that a particular set can generate when k increases by 1, a pattern emerges.

$$\left.\begin{matrix} [1\ 2\ 3\ 4\ .\ .] \\ [1\ 2\ 3\ 5\ .\ .] \\ [1\ 2\ 3\ 6\ .\ .] \\ [1\ 2\ 3\ 7\ .\ .] \end{matrix}\right\} 4$$

The matrix [1 2 3 4 . .], for c_5, would generate matrices which all start [1 2 3 4 . .].

In general, for c_5 this matrix generates the set of matrices

$\left[1\ \ 2\ \ 3\ \ 4\ \ p\ \ .\ \ .\right],\ 5\le p\le 9\big\}\rightarrow 5$ matrices.

The matrix $\left[1\ \ 2\ \ 3\ \ 5\ \ .\ \ .\right]$ gives the set of matrices

$\left[1\ \ 2\ \ 3\ \ 5\ \ p\ \ .\ \ .\right],\ 6\le p\le 9\big\}\rightarrow 4$ matrices.

The matrix $\left[1\ \ 2\ \ 3\ \ 6\ \ .\ \ .\right]$ gives the set of matrices

$\left[1\ \ 2\ \ 3\ \ 6\ \ p\ \ .\ \ .\right],\ 7\le p\le 9\big\}\rightarrow 3$ matrices.

The matrix $\left[1\ \ 2\ \ 3\ \ 7\ \ .\ \ .\right]$ gives the set of matrices

$\left[1\ \ 2\ \ 3\ \ 7\ \ p\ \ .\ \ .\right],\ 8\le p\le 9\big\}\rightarrow 2$ matrices.

It can be seen from this that a set of 4 matrices generates:

- A set of 5 matrices
- A set of 4 matrices
- A set of 3 matrices
- A set of 2 matrices

This pattern can be generalized:

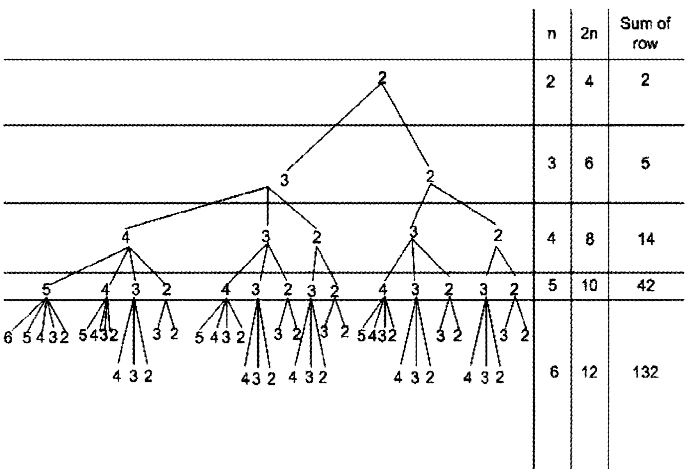
A set of w matrices generates:

- A set of $w + 1$ matrices
- A set of w matrices
- A set of $w - 1$ matrices

...

- A set of 4 matrices
- A set of 3 matrices
- A set of 2 matrices

These generated sets can be viewed as sequences; for example, a set containing 4 values will generate a sequence of sets containing 5, 4, 3 and 2 members. Now this is starting to become interesting. If we can get from the first set of 2 matrices (c_2), to the next sequence, and from the next sequence to the sequence after that, we could potentially find the number of matrices in any sequence. Then all we would have to do is to add up the total number of matrices in all the sequences with matrices of length n, and we would find our solution. However, you might realize that this would take a long time. I have done it in the diagram below for $2n \le 12$, but it is already becoming quite hard work.



The following table details the number of times each particular term of a sequence appears in $2n$. For example, in the diagram, there are 14 '2's in the $2n = 12$ row. If the sum of the coefficients in the table with their relative multipliers along the top row is taken for a particular row, it will generate the number of permutations possible for that row. For example, for $2n = 6$, we add up:

$$1 \times 2 + 1 \times 3 = 2 + 3 = 5,$$

so there are 5 possibilities for 6 children in a photo. Check this result by writing the possibilities out if you want.

n	2*n*	x2	x3	x4	x5	x6	x7	x8	x9
2	**4**	1							
3	**6**	1	1						
4	**8**	2	2	1					
5	**10**	5	5	3	1				
6	**12**	14	14	9	4	1			
7	**14**	42	42	28	14	5	1		
8	**16**	132	132	90	48	20	6	1	
9	**18**	429	429	297	165	75	27	7	1

We have already seen that a set consisting of w matrices will generate a sequence:

$$w + 1,\ w, w - 1, w - 2, \ldots, 4, 3, 2.$$

Adding an extra column to the table for the total number of permutations gives us an insight into how to proceed:

n	2*n*	x2	x3	x4	x5	x6	x7	x8	x9	Permutations
2	**4**	1								2
3	**6**	1	1							5
4	**8**	2	2	1						14
5	**10**	5	5	3	1					42
6	**12**	14	14	9	4	1				132
7	**14**	42	42	28	14	5	1			429
8	**16**	132	132	90	48	20	6	1		1430
9	**18**	429	429	297	165	75	27	7	1	4862

It can be seen clearly that the '2' coefficient for the row '$n + 2$' is equal to the number of permutations for the row 'n', which can be generalized for all n.

So we proceed to find a recurrence relation to get from one row to the next. The coefficient of a particular multiplier can be denoted by $r_n(z)$ where z is the multiplier, e.g. $r_9(9) = 1$, $r_9(8) = 7$, $r_9(7) = 27$.

For all $n > 2$

$$r_n(n) = r_{n-1}(n - 1) = 1$$

$$r_n(n - 1) = r_{n-1}(n - 1) + r_{n-1}(n - 2) = n - 2$$

$$r_n(n - 2) = r_{n-1}(n - 1) + r_{n-1}(n - 2) + r_{n-1}(n - 3) = n - 2 + r_{n-1}(n - 3).$$

We can continue like this to find relations for all of the cells in the table; however, it doesn't help us very much unless we can find a simpler way of writing it.

$$\begin{aligned} r_n(n-2) &= (n - 2) + r_{n-1}(n - 3) \\ &= (n - 2) + (n - 3) + r_{n-2}(n-4) \\ &= (n - 2) + (n - 3) + (n - 4) + r_{n-3}(n - 5) \\ &\ \ \ldots \\ &= (n - 2) + (n - 3) + (n - 4) + (n - 5) + \ldots + 3 + 2 \end{aligned}$$

$$r_n(n-2) = \sum_{2}^{n-2} k.$$

So, it turns out that we can express the coefficient of $(n-2)$ in the row n as simply a sum from 2 to $(n-2)$.

We can use this to produce an expression for the coefficient of $(n-3)$ as follows:

$$r_n(n-3)=\sum_{2}^{n-2}k+r_{n-1}(n-4)$$

$$r_n(n-3)=\sum_{2}^{n-2}k+r_{n-1}(n-3)+r_{n-2}(n-5)$$

$$r_n(n-3)=\sum_{2}^{n-2}k+\sum_{2}^{n-3}k+r_{n-2}(n-5)$$

$$r_n(n-3)=\sum_{2}^{n-2}k+\sum_{2}^{n-3}k+\sum_{2}^{n-4}k+r_{n-3}(n-6)$$

$$r_n(n-3)=\sum_{2}^{n-2}k+\sum_{2}^{n-3}k+\sum_{2}^{n-4}k+\ldots+\sum_{2}^{3}k$$

$$r_n(n-3)=\sum_{a_1=3}^{n-2}\sum_{2}^{a}k.$$

We can produce relations for all of the other coefficients as well.

$$r_n(n-4)=\sum_{a_2=4}^{n-2}\sum_{a_1=3}^{a_2}\sum_{2}^{a}k$$

...

$$r_n(2)=\sum_{a_{n-3}=n-1}^{n-2}\ldots\sum_{a_2=4}^{a_3}\sum_{a_1=3}^{a_2}\sum_{2}^{a}k.$$

So this has led us to a formula for the number of permutations for $2n$ pupils in the school photograph as:

$$P_{2n}=r_{n+2}(2)=\sum_{a_{n-3}=n-1}^{n}\ldots\sum_{a_2=4}^{a_3}\sum_{a_1=3}^{a_2}\sum_{k=2}^{a_1}k.$$

This can be simplified even further, by employing the following summation formulae:

$$\sum_{1}^{n}k=\frac{1}{2!}n(n+1)$$

$$\sum_{a_1=1}^{n}\sum_{1}^{a_1}k=\frac{1}{3!}n(n+1)(n+2)$$

...and so on.

By using these multiple times, and keeping track of all of the ns and other bits and pieces, we can produce a really nice formula for the total number of permutations.

Let's start by simplifying the number of permutations for $n=3$, where $2n=6$.

Let us consider:

$$\sum_{a_1=1}^{3}\sum_{1}^{a_1}k-\sum_{a_1=3}^{3}\sum_{2}^{a_1}k=\left(\sum_{1}^{1}k+\sum_{1}^{2}k+\sum_{1}^{3}k\right)-\left(\sum_{2}^{3}k\right).$$

By changing the lower value on the initial Σ from 1 to 2, and changing the lower value on the second Σ from 2 to 3, we have effectively:

1. Subtracted 1, as $\sum_{1}^{3}k-\sum_{2}^{3}k=1.$
2. Subtracted another 3, as the $\sum_{1}^{2}k=1+2=3$ has been removed as well.
3. Subtracted another 1, as the $\sum_{1}^{1}k=1.$

So,

$$\sum_{a_1=1}^{3}\sum_{1}^{a_1}k-\sum_{a_1=3}^{3}\sum_{2}^{a_1}k=3+2.$$

Therefore

$$\sum_{a_1=3}^{3}\sum_{2}^{a_1}k=\frac{1}{3!}\cdot3\cdot(3+1)(3+2)-(3+2)$$

$$=\frac{1}{3!}(3-2)(3+2)(3+3).$$

Extending this for any n

$$P_{2n}=r_{n+2}(2)=\sum_{a_{n-3}=n-1}^{n}\ldots\sum_{a_2=4}^{a_3}\sum_{a_1=3}^{a_2}\sum_{k=2}^{a_1}k$$

$$=\frac{1}{n!}(n-(n-1))(n+n)(n+n-1)\ldots(n+2).$$

This can be expressed in factorial form as

$$\frac{(n-(n-1))(n+n)(n+n-1)\ldots(n+2)(n+1)\ldots3\cdot2\cdot1}{(n+1)\ldots3\cdot2\cdot1}\frac{1}{n!}$$

$$=\frac{(n+n-1)(n+n)!}{n!(n+1)!}=\frac{(2n)!}{n!(n+1)!}$$

This formula in fact holds for all n, which yields a fairly simple, closed solution to the problem.

$$\boxed{P_{2n}=\frac{(2n)!}{n!(n+1)!}}$$

Keywords: Problem solving.

Author
Julian Parmar (student), Haberdashers' Aske's Boys' School, Elstree, Herts WD6 3AF.
e-mail: **par510@habsboys.org.uk**